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## Final Report for AOARD Grant 124097

### Control Strategies for Guided Collective Motion

30 January 2015

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#### Abstract

Guided collective motion of cooperating vehicles was addressed in this project through the use of heterogeneous guidance gains. Two different approaches were examined for this purpose. In the first we assumed all-to-all communication between agents and proposed strategies that allowed the agent swarm to converge to specified points in the environment. Several strategies were tried out for this purpose using balanced and splay formation objectives. The second approach was to exploit the cyclic pursuit strategy where the connections are with one leader and one follower and a similar heterogeneous gains and hierarchical structure was exploited to achieve convergence at arbitrary points in space.

#### Introduction

Collective motion of particles is an area that has attracted the attention of the multi-agent cooperative control community of researchers. Designing cooperative control strategies that exploit symmetry and homogeneity of the system to achieve collective motion under global or partial communication network was the main challenge addressed by these researchers. However, the property of homogeneity is never realized in a real system. There exist results that show the robustness of these control strategies even under heterogeneity. However, the heterogeneity that these works consider is related to the heterogeneity in the dynamics of the systems.

In this proposal, we examined this issue of heterogeneity from the point of view of introducing heterogeneity in the control strategy in order to achieve some desired behavior. For example, collective motion aims to achieve consensus in terms of a target point about which the particles (or vehicles – UAVs, AUVs, or robots – with simple dynamics) achieve a equally spaced circular motion, or consensus in terms of a common direction of parallel movement. However, in any real applications it is necessary to specify a desired point about which this collective motion should occur or specify a path along which the parallel motion to occur. In the literature, the task of guiding collective motion has been done by introducing the angle of movement in the parallel motion control strategy and then switching off the parallel motion strategy and switching on the circular motion strategy when the formation approximately reaches the desired target point. In another approach this has been achieved by introducing a flow field.

In the proposed work we aim to achieve these requirements of target point surveillance by circular motion or parallel motion along a desired path by introducing heterogeneity in the system in terms of non-homogeneous control gains. The advantage is that explicit commands to the vehicles about the intended target point or path can be avoided. Instead, controller gains can be made to serve the same objective while camouflaging their intention from an adversary.

## Techniques and Approach

We have used two different paradigms to achieve the task of collective motion under heterogeneous guidance gains. One of them uses an all-to-all communication framework between agents and attempts to achieve point convergence as well as collective motion in balanced and splay formations about a target point of interest. The second approach uses a cyclic pursuit paradigm that requires each agent to have a leader and a follower. This has been further generalized in terms of heterogeneous guidance gains, multiple leaders and followers, hierarchical structures, and single and double integrator dynamics. The major results for these have been described below.

## Results and Discussion

In the all-to-all communication framework the following results have been obtained.

Collective Motion with Heterogeneous Controllers: In this work, we study the collective motion of individually controlled planar particles when they are coupled through heterogeneous controller gains. Two types of collective formations, synchronization and balancing, are described and analyzed under the influence of these heterogeneous controller gains. These formations are characterized by the motion of the centroid of the group of particles. In synchronized formation, the particles and their centroid move in a common direction, while in balanced formation the movement of particles possesses a fixed location of the centroid. We show that, by selecting suitable controller gains, these formations can be controlled significantly to obtain not only a desired direction of motion but also a desired location of the centroid. We present the results for  $N$ -particles in synchronized formation, while in balanced formation our analysis is confined to two and three particles.

Stabilization of Balanced Circular Motion: In this work, we study the collective motion of a group of  $N$  ( $>=2$ ) identical agents trying to achieve a circular formation centered at a desired location, which is fixed. A circular formation is characterized by the motion of all agents around the same circle in the same direction. To solve this problem, we propose a planar motion model that incorporates two control inputs. One of the control inputs is chosen independently and the other control input is decided by using the composite Lyapunov function. We show that the desired location of the center of this circular formation, which is fixed, is obtained by directing the centroid of the group of agents to that desired location. This leads to a collective formation of all the agents, known as balanced circular formation. The theoretical findings are supported by simulations.

In the cyclic pursuit framework the following results have been obtained.

Reachability with double integrator dynamics: A new law is proposed which guarantees stability for agents with double integrator dynamics. An algorithm is proposed which enables rendezvous of the agents at any desired point in the two-dimensional space. The gains corresponding to each agent are different and, along with their initial velocities, are considered to be the decision variables.

Global Reachability and Target Capturability: Global reachability of agents under a cyclic pursuit framework has been discussed and a potential application of the expansion in reachable set has been pointed out with respect to capture of a moving target. Agents with double integrator dynamics have also been considered.

General Hierarchical Pursuit: In this work, a variant of cyclic pursuit, called hierarchical cyclic pursuit, has been considered. The underlying information flow graph that connects the agents is hierarchical in nature. In prior work, the gains corresponding to each agent were considered equal. This work generalizes the existing results in the form of heterogeneity in the gains, and presents some results on the reachability of the agents in hierarchical cyclic pursuit. It has been shown that the existing results may be obtained as special cases of the one considered here.

Deviated Cyclic Pursuit: This work addresses and analyses deviated linear cyclic pursuit in which an

agent pursues its leader with an angle of deviation. The sufficient conditions for the stability of such systems are presented in this paper along with the derivation of the reachable set, which is a set of points where the agents may converge asymptotically. Both continuous time and discrete time cases are considered

Synchronous and Asynchronous Cyclic Pursuit: In the existing literature, cyclic pursuit has been mostly considered in the continuous time framework and all the results have been derived based on the continuous time model. However, in most practical applications and implementations, the systems under consideration are of discrete time in nature. This is because sampled data has to be used for computation purposes. The discrete time model of cyclic pursuit calls for some new approaches and techniques because the synchronization between the agents determines the stability of the system. This work provides some stability results for heterogeneous cyclic pursuit in the discrete time domain by considering both synchronous and asynchronous frameworks.

Synchronized, Balanced and Splay Phase Arrangements: This work proposes a design methodology to stabilize collective circular motion of a group of  $N$ -identical agents moving at unit speed around individual circles of different radii and different centers. The collective circular motion studied in this paper is characterized by the clockwise rotation of all agents around a common circle of desired radius as well as center, which is fixed. Our interest is to achieve those collective circular motions in which the phases of the agents are arranged either in synchronized, in balanced or in splay formation. In synchronized formation, the agents and their centroid move in a common direction while in balanced formation, the movement of the agents ensures a fixed location of the centroid. The splay state is a special case of balanced formation, in which the phases are separated by multiples of  $2\pi/N$ . We derive the feedback controls and prove the asymptotic stability of the desired collective circular motion by using Lyapunov theory and the LaSalle's Invariance principle.

Achieving a Stationary or Moving Centroid: In this work, we study the problem of a formation of agents trying to achieve a desired collective centroid, which might be stationary or moving. The stabilization of the collective centroid to the fixed desired location leads to a balanced formation of the agents about that point. Similarly, the centroid of the system of agents may be needed to move along a certain given trajectory. For this, the centroid of the formation must converge to the desired trajectory. To solve this problem, we propose a planar motion model that explicitly incorporates a controlled parameter. We also provide the results for the synchronized formation where the agents, along with that of their centroid, move in a common velocity direction. Simulation results are presented to support the theoretical findings.

Achieving Desired Angular Frequency: In this work, we consider two types of collective formations: (i) synchronized circular formation and (ii) balanced circular formation. These formations are characterized by the motion of the centroid of the group of agents. In synchronized circular formation, the agents and their centroid rotate on their individual circles in a common direction, while in balanced circular formation, the circling of the agents gives rise to a fixed location of the centroid. We show that the agents, having heterogeneously distributed initial angular frequencies, can be made to stabilize to synchronized and balanced circular formations at a desired angular frequency. Also, the radius of these circular formations can be controlled significantly by controlling their desired angular frequency. We further extend the analysis to prove the stabilization of balanced circular formation at the desired angular frequency, and to achieve a desired location of the centroid.

**List of Publications and Significant Collaborations that resulted from your AOARD supported project:**

a) Papers published in peer-reviewed journals

*Papers are being submitted to journals. A list is given below.*

b) Papers published in peer-reviewed conference proceedings

1. A. Jain and D. Ghose, Collective Behavior with Heterogeneous Controllers, *Proceedings of the American Control Conference*, Washington, DC, USA, June 17-19, 2013, pp. 4636-4641.
2. A. Jain, D. Ghose, P.P. Menon, Stabilization of Balanced Circular Motion About a Desired Center," *Proceedings of the Advances in Control and Optimization of Dynamical Systems*, ACODS-2014, IIT Kanpur, India, 2014, pp. 656-661.
3. A. Jain and D. Ghose, Stabilization of Collective Motion in Synchronized, Balanced and Splay Phase Arrangements on a Desired Circle, Accepted in the American Control Conference, July 2015, Chicago, USA.
4. D. Mukherjee and D. Ghose, Reachability of Agents with Double Integrator Dynamics in Cyclic Pursuit, *52nd IEEE Conference on Decision and Control*, December 10-13, 2013. Florence, Italy, pp. 5397-5402.

c) Papers published in non-peer-reviewed journals and conference proceedings

*None*

d) Conference presentations without papers

*None*

e) Manuscripts submitted to journals, or under preparation, but not yet published

1. A. Jain and D. Ghose, Stabilization of Synchronized and Balanced Collective Motion With Heterogeneous Controllers,
2. A. Jain, D. Ghose and P.P. Menon Achieving a Desired Stationary or Moving Collective Centroid by a Formation of Agents.
3. A. Jain and D. Ghose Stabilization of the Synchronized and Balanced Circular Motions at Desired Angular Frequency.
4. D. Mukherjee and D. Ghose: Global Reachability and Target Capturability using Cyclic Pursuit.
5. D. Mukherjee and D. Ghose: Generalized Hierarchical Cyclic Pursuit
6. D. Mukherjee and D. Ghose: Deviated Linear Cyclic Pursuit
7. D. Mukherjee and D. Ghose; On Synchronous and Asynchronous Heterogeneous Cyclic Pursuit

f) Provide a list any interactions with industry or with Air Force Research Laboratory scientists or significant collaborations that resulted from this work.

Dr. Prathyush P. Menon, University of Exeter, UK: The AOARD project has facilitated interactions with Dr. Menon on designing algorithms for contaminant detection and spread using collective motion algorithms.

Dr. K.N. Kaipa, University of Maryland; The AOARD project has helped us to build collaborative connections with Dr. Kaipa to exploit teams of robots by implementing swarm intelligence algorithms in them.

**Attachments:** Publications a), b) and c) listed above if possible.

*The publications are attached with this report*

**DD882:** As a separate document, please complete and sign the inventions disclosure form.

*Done*

# Collective Behavior with Heterogeneous Controllers

Anoop Jain<sup>1</sup> and Debasish Ghose<sup>2</sup>

**Abstract**—In this paper, we study the collective motion of individually controlled planar particles when they are coupled through heterogeneous controller gains. Two types of collective formations, synchronization and balancing, are described and analyzed under the influence of these heterogeneous controller gains. These formations are characterized by the motion of the centroid of the group of particles. In synchronized formation, the particles and their centroid move in a common direction, while in balanced formation the movement of particles possess a fixed location of the centroid. We show that, by selecting suitable controller gains, these formations can be controlled significantly to obtain not only a desired direction of motion but also a desired location of the centroid. We present the results for  $N$ -particles in synchronized formation, while in balanced formation our analysis is confined to two and three particles.

## I. INTRODUCTION

Formation control of Unmanned Aerial Vehicles (UAVs) and Autonomous Underwater Vehicles (AUVs) has emerged as a growing area of research because of its applications to various missions such as surveillance, search and data collection [1], [2]. The particle model [3] of these vehicles provides a simple design methodology to analyze their collective motion [4]-[9]. This paper deals with two types of collective motions: synchronization and balancing, for all-to-all heterogeneously coupled particles. Synchronization refers to the situation when, at all times, the particles and their centroid, which is the average position of the group of particles, have a common velocity direction. On the other hand, balancing refers to the situation in which the particles move in such a way that their centroid remains fixed. In balanced formation, the centroid is called the convergence point to distinguish its motion in synchronized formation.

The present work is inspired by the problem addressed in [10], which proposed a steering control law to stabilize synchronized and balanced formations. The proposed control law operates with homogeneous controller gains for individually controlled particles. This type of homogeneous coupling among agents limits its applications in the field of aerial and underwater vehicles. As an extension in this paper, it is assumed that the particles are heterogeneously coupled, and that the controller gains can be deterministically varied. In [11], the authors have shown that the meeting point (rendezvous) of multiple agents in linear cyclic pursuit can be decided by selecting proper heterogeneous gains. The

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analytical results that relate the heterogeneous controller gains to the direction of movement of the agents when the system is unstable, are derived in [12]. A more general case, where the speeds and controller gains for the different agents may vary, is discussed in [13]. In the present paper, some interesting possibilities regarding the synchronized and balanced formations based on the assumption of heterogeneous coupling are presented.

## II. ANALYSIS OF HETEROGENEOUS CONTROLLER GAINS

In the literature [10], [14], [15], collective dynamics of multiple autonomous agents is studied through the particle model in which each individual is represented by a particle having unit mass. In [10], the authors introduce a model of steering particles moving with unit speed as:

$$\dot{r}_k = e^{i\theta_k}; \quad \dot{\theta}_k = u_k, \quad k = 1, \dots, N \quad (1)$$

Here, the position of the  $k$ -th particle is  $r_k = x_k + iy_k$ , while the velocity of the  $k$ -th particle is  $v_k \triangleq e^{i\theta_k} = \cos \theta_k + i \sin \theta_k$ , where,  $\theta_k$  denotes the orientation of the (unit) velocity vector of the  $k$ -th particle from the positive  $x$ -axis. The orientation of the velocity vector is also referred to as the phase of the particle, which is motivated from the problem of achieving synchronization in coupled oscillators [16], [17].

The steering control  $u_k$  is a feedback control law for the  $k$ -th particle. If, for all  $k = 1, \dots, N$ , the control  $u_k$  is identically zero, then each particle travels in a straight line in its initial direction  $\theta_k(0)$ . If, on the other hand, for all  $k = 1, \dots, N$ ,  $u_k \triangleq \omega_0$  is constant but not zero, then each particle travels on a circle with radius  $|\omega_0|^{-1}$ .

The control of the average linear momentum of the group plays an important role in the controller design methodology [10]. The average linear momentum  $p_\theta$  of a group of particles satisfying (1) is,

$$p_\theta \triangleq \frac{1}{N} \sum_{k=1}^N \dot{r}_k = \frac{1}{N} \sum_{k=1}^N e^{i\theta_k} \quad (2)$$

The magnitude  $|p_\theta|$  satisfies  $0 \leq |p_\theta| \leq 1$ . In the particle model defined by (1), synchronization of the phases corresponds to a parallel formation where all particles move in the same direction, which occurs when  $|p_\theta| = 1$ . In contrast, balancing of the phases corresponds to collective motion around a fixed point, which occurs when  $|p_\theta| = 0$ . In [10], a gradient based control law  $u_k$  is designed by optimizing the potential function  $U(\theta)$ , given by

$$U(\theta) = (N/2) |p_\theta|^2 \quad (3)$$

which is maximized by synchronized phase arrangements and minimized by balanced phase arrangements of a group

of moving particles. Most of the previous work uses the same controller gain  $K$  for all the particles, whereas we generalize the system by using different coupling gains  $K_k$  for different particles. The gradient control law is hence given as,

$$u_k = -K_k \text{grad}U = -K_k (\partial U / \partial \theta_k), \quad k = 1, \dots, N \quad (4)$$

Eq. (4) can be simplified as

$$u_k = -(K_k/N) \sum_{j=1}^N \sin(\theta_j - \theta_k) \quad (5)$$

This control law controls the phases of the particles and defines a phase model as

$$\dot{\theta}_k = \omega_0 + u_k \quad (6)$$

where,  $\omega_0$  is the initial angular velocity of each particle. Eq. (5) is known as the Kuramoto model and widely studied in the literature [19]-[21]. The following theorem ensures the asymptotic stability of the synchronized and the balanced formations for heterogeneous controller gains  $K_k$  for all  $k = 1, \dots, N$ .

*Theorem 1:* For the gradient control law (4), all the solutions of the phase model (6), converge to the critical set of  $U(\theta)$ , which is the set of all points where the gradient of  $U(\theta)$  is zero. If  $\sum_{k=1}^N K_k T_k < 0$ , then all the synchronized phase

arrangements are asymptotically stable and if  $\sum_{k=1}^N K_k T_k > 0$ , then the balanced phase arrangements are asymptotically stable, where  $T_k = \left\{ \sum_{j=1}^N \sin(\theta_j - \theta_k) \right\}^2$ . Moreover, the gains  $K_k$ , provide a sufficient condition for the synchronized (when  $K_k < 0, \forall k$ ) and balanced (when  $K_k > 0, \forall k$ ) phase arrangements.

*Proof:* The time derivative of the potential function  $U(\theta)$  is given as

$$\dot{U}(\theta) = \sum_{k=1}^N \frac{\partial U}{\partial \theta_k} \dot{\theta}_k \quad (7)$$

Using Eqs. (4), (5) and (6),  $\dot{U}(\theta)$  becomes

$$\dot{U}(\theta) = -\frac{1}{N^2} \sum_{k=1}^N K_k T_k \quad (8)$$

In the above equation, there is no term containing  $\omega_0$ . It is because of the orthogonal property of the  $\text{grad}U$  vector with the vector  $\mathbf{1}$ , where  $\mathbf{1} \triangleq (1 \ 1 \ \dots \ 1)^T \in \mathbb{R}^N$ . It means that  $\langle \text{grad}U, \mathbf{1} \rangle = 0$ , where,  $\text{grad}U = \left[ \frac{\partial U}{\partial \theta_1}, \dots, \frac{\partial U}{\partial \theta_N} \right]^T$  and  $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$ , represents the inner product of real vectors  $\mathbf{x}$  and  $\mathbf{y}$ . Eq. (8) shows that the potential function  $\dot{U}(\theta)$  is a monotonically increasing or decreasing function according to the sign of the term  $\sum_{k=1}^N K_k T_k$ . Hence, if for all  $k = 1, \dots, N$ ,

$$\sum_{k=1}^N K_k T_k > 0 \Rightarrow \dot{U}(\theta) < 0 \quad (9)$$

then it implies that the potential function  $U(\theta)$  approaches its least value when all the gain parameters are positive. The minimum value of  $U(\theta)$  occurs at  $|p_\theta| = 0$ . The condition (9) indicates the situation when the average position of all the particles is fixed, that is, the particles form a balanced state. Similarly, for all  $k = 1, \dots, N$

$$\sum_{k=1}^N K_k T_k < 0 \Rightarrow \dot{U}(\theta) > 0 \quad (10)$$

implies that the potential function  $U(\theta)$  approaches its peak value, when all the gain parameters are negative. The maximum value of  $U(\theta)$  occurs at  $|p_\theta| = 1$ . The condition (10) indicates the situation when all the particles show a synchronized behavior in which all the particles have the same velocity direction at any instant of time. Note that

$$T_k \geq 0, \quad k = 1, \dots, N.$$

and equality holds for the trivial solution when all the particles are already in the critical set [2]. Therefore, for nontrivial solutions ( $T_k > 0$ ),  $K_k < 0$  or  $K_k > 0$ , for all  $k = 1, \dots, N$  is, respectively, a sufficient condition for the synchronized and the balanced phase arrangements. This completes the proof. ■

*Corollary 1:* The desired phase arrangements, for a special two particle case when  $N = 2$ , are realizable for combined positive and negative values of the gains  $K_k$ .

*Proof:* This extra condition on the controller gain parameters can be derived from the following expressions for  $N = 2$ , written by using (5), (6) and (8) as (for  $i = 1, 2$ )

$$\dot{\theta}_i = \omega_0 - \frac{K_i}{2} \sum_{j=1}^2 \sin(\theta_j - \theta_i) \quad (11)$$

$$\dot{U}(\theta) = -\frac{1}{2^2} (K_1 + K_2) [\sin^2(\theta_2 - \theta_1)] \quad (12)$$

So,

$$K_1 + K_2 > 0 \Rightarrow \dot{U}(\theta) < 0 \quad (13)$$

and

$$K_1 + K_2 < 0 \Rightarrow \dot{U}(\theta) > 0 \quad (14)$$

It is clear that both positive and negative values of the gains  $K_1$  and  $K_2$  can together satisfy the conditions (13) and (14). ■

### III. BALANCED FORMATION

Balanced formation corresponds to the motion of particles around a fixed point, which is the average position (centroid) of the group, and occurs when the average linear momentum ( $p_\theta$ ) of the group is zero. The average position of the particle group is given by

$$R \triangleq \frac{1}{N} \sum_{k=1}^N r_k \Rightarrow \dot{R} = \frac{1}{N} \sum_{k=1}^N \dot{r}_k = \frac{1}{N} \sum_{k=1}^N e^{i\theta_k} \quad (15)$$

Therefore, the average linear momentum ( $p_\theta$ ) is the velocity  $\dot{R}$  of the average position point. In the literature,  $\dot{R}$  is referred

to as the order parameter [18], and balanced formation can be achieved when

$$\dot{R} = 0 \Rightarrow \sum_{k=1}^N \cos \theta_k = 0 \quad \text{and} \quad \sum_{k=1}^N \sin \theta_k = 0 \quad (16)$$

**Theorem 2:** Balanced formation of two and three particles ( $N = 2$  or  $3$ ) with phase model (6), respectively, occurs iff their final velocity directions after stabilization, are at angular separations of  $\pi$  and  $\frac{2\pi}{3}$  radians. Moreover, the final velocity direction of one of the particles, is given by

$$\theta_f = \omega_0 t + \sum_{i=1}^N \frac{1}{K_i} \left( \theta_{i0} - \frac{2(i-1)\pi}{N} \right) \Bigg/ \sum_{i=1}^N \frac{1}{K_i} \quad (17)$$

*Proof:* For the special case when  $N = 2$ , let the final velocity directions of the particles, stabilized to balanced formation, be related as

$$\theta_{1f} = \theta_{2f} - \sigma = \theta_f \quad (\text{say}) \quad (18)$$

where,  $\sigma \in [0, 2\pi)$  is the angular separation between the particles. After stabilization to balanced formation, the angular rates  $\dot{\theta}_i$  of their motions should have settled down to  $\omega_0$ . Substituting (18) into (6), we conclude that  $\sigma = \pi$ . This result can also be verified from (16). Likewise, let, for the case when  $N = 3$ , the final velocity directions of the particles, stabilized to balanced formation, be related as

$$\theta_1 = \theta_f, \quad \theta_2 = \theta_f + \psi_1, \quad \theta_3 = \theta_f + \psi_2 \quad (19)$$

Here,  $\psi_1, \psi_2 \in [0, 2\pi)$  and are the angular separations between the final velocity directions of particles 1 and 2, and 1 and 3, respectively. From (16), (19),

$$\cos(\theta_f + \psi_1) + \cos(\theta_f + \psi_2) = -\cos \theta_f \quad (20)$$

$$\sin(\theta_f + \psi_1) + \sin(\theta_f + \psi_2) = -\sin \theta_f \quad (21)$$

Squaring and adding (20) and (21), we get  $\cos(\psi_2 - \psi_1) = -1/2$ , which implies that  $\psi_2 - \psi_1 = 2n\pi + 2\pi/3, \forall n \in \mathbb{Z}$ . We also get the following expressions by using (6), which follows from the fact that the angular rates of the particles motions are  $\omega_0$  when they stabilize to a balanced formation.

$$\begin{cases} \sin \psi_1 + \sin \psi_2 = 0 \\ -\sin \psi_1 + \sin(\psi_2 - \psi_1) = 0 \\ -\sin \psi_2 + \sin(\psi_1 - \psi_2) = 0 \end{cases} \quad (22)$$

Putting  $\psi_2 - \psi_1 = 2n\pi + 2\pi/3$  in (22), we get  $\psi_1 = 2n_1\pi + 2\pi/3$  and  $\psi_2 = 2n_2\pi + 4\pi/3, \forall n_1, n_2 \in \mathbb{Z}$ . To satisfy  $\psi_1, \psi_2 \in [0, 2\pi)$ , we choose  $n_1 = n_2 = 0$  to get  $\psi_1 = 2\pi/3$  and  $\psi_2 = 4\pi/3$ . These results show that the final velocity directions of the particles (for  $N = 3$ ) are at  $2\pi/3$  radians angular separations. It is straightforward to prove the sufficiency condition and hence this proof is omitted. Now, the final velocity direction  $\theta_f$  can be found easily. From (5) and (6), we conclude that

$$\sum_{i=1}^N \frac{\dot{\theta}_i}{K_i} = \sum_{i=1}^N \frac{\omega_0}{K_i} \Rightarrow \sum_{i=1}^N \frac{\theta_i}{K_i} = \sum_{i=1}^N \left( \frac{\omega_0 t + \theta_{i0}}{K_i} \right) \quad (23)$$

Here,  $\theta_{i0}$  is the initial velocity direction of the  $i$ -th particle. Substituting (18) and (19) (with defined values of

$\sigma, \psi_1$  and  $\psi_2$ ), respectively, into (23), the final velocity direction  $\theta_f$ , for  $N = 2, 3$ , is obtained as specified in (17). Here,  $t$  is the simulation time. This completes the proof. ■

For the two particle case ( $N = 2$ ), one can also obtain explicit expressions for the velocity directions as a function of time. From Eqs. (5), (6) and (23) one can form a differential equation in terms of  $\theta_1$  as

$$\dot{\theta}_1 + \frac{K_1}{2} \sin(K_2 c_0 + \alpha(\omega_0 t - \theta_1)) - \omega_0 = 0 \quad (24)$$

which can be solved easily by the method of separation of variables to yield

$$\theta_1 = \omega_0 t + (1/\alpha) \left[ K_2 c_0 - 2 \arctan \left\{ (1/\phi e^{-\lambda t}) \right\} \right] \quad (25)$$

$$\theta_2 = \omega_0 t + (1/\beta) \left[ K_1 c_0 + 2 \arctan \left\{ (1/\phi e^{-\lambda t}) \right\} \right] \quad (26)$$

where,  $c_0 = ((\theta_{10}/K_1) + (\theta_{20}/K_2))$ ,  $\alpha = (K_1 + K_2)/K_1$ ,  $\beta = (K_1 + K_2)/K_2$ ,  $\lambda = (K_1 + K_2)/2$  and  $\phi = \cot(\theta_{20} - \theta_{10})/2$  are constants.

These results show how heterogeneous controller gains affect the particle's velocity directions. It is clear from (25) and (26) that for a large time  $t$ , the velocity directions stabilize to constant values. So, in the steady-state,  $\theta_1 = \omega_0 t + (1/\alpha) [K_2 c_0 \pm \pi]$  and  $\theta_2 = \omega_0 t + (1/\beta) [K_1 c_0 \pm \pi]$ . These results imply that  $|\theta_1 - \theta_2| = \pi$ . Hence, the difference between the final velocity directions is  $\pi$ , as desired.

#### IV. CONVERGENCE POINT FOR $N = 2$

The centroid of the particle group is stabilized to a fixed point when the particles form a balanced formation. This fixed point is named as the convergence point of the system. This convergence point can be adjusted by the heterogeneous controller gains  $K_k$ . The rate of change of the convergence point can be written from (15) as

$$\dot{R} = \dot{x}_c + i\dot{y}_c \triangleq \frac{1}{N} \sum_{k=1}^N e^{i\theta_k} \quad (27)$$

where,  $x_c$  and  $y_c$  are the abscissa and the ordinate of the point of convergence.

**Lemma 1:** The convergence point, for  $N = 2$ , approaches the initial centroid for large positive value of  $K_1 + K_2$ , where  $K_1$  and  $K_2$  are heterogeneous coupling gains of two particles.

*Proof:* For two particles ( $N = 2$ ), we can write

$$\dot{x}_c = \frac{1}{2} [\cos \theta_1 + \cos \theta_2]; \quad \dot{y}_c = \frac{1}{2} [\sin \theta_1 + \sin \theta_2] \quad (28)$$

Integrating (28), we get

$$\begin{aligned} x_c(t) - x_{c0} &= (1/2) \int_0^t \{\cos \theta_1 + \cos \theta_2\} d\tau \\ y_c(t) - y_{c0} &= (1/2) \int_0^t \{\sin \theta_1 + \sin \theta_2\} d\tau \end{aligned} \quad (29)$$

where,  $(x_{c0}, y_{c0})$  are the co-ordinates of the initial centroid. We can easily compute the above integrals. By using (25)

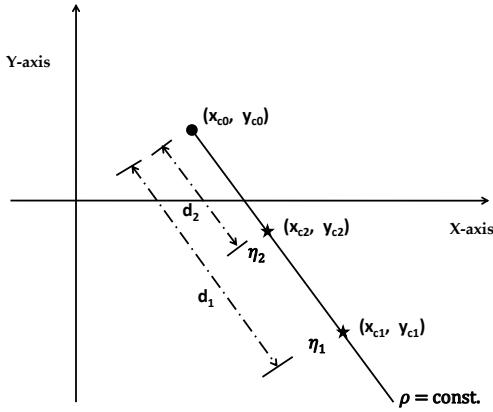


Fig. 1. Locus of the convergence point for constant  $\rho$

and (26), we can write

$$x_c(t) - x_{c0} = \int_0^t (\phi e^{-\lambda \tau} \cos \Theta) / (\sqrt{1 + \phi^2 e^{-2\lambda \tau}}) d\tau \quad (30)$$

$$y_c(t) - y_{c0} = \int_0^t (\phi e^{-\lambda \tau} \sin \Theta) / (\sqrt{1 + \phi^2 e^{-2\lambda \tau}}) d\tau \quad (31)$$

where,  $\Theta(\tau) = [\omega_0 \tau + \delta - \phi_0 \arctan(1/\phi e^{-\lambda \tau})]$ ,  $\phi_0 = \left(\frac{1}{\alpha} - \frac{1}{\beta}\right) = \left(\frac{K_1 - K_2}{K_1 + K_2}\right)$  and constant  $\delta = \left(\frac{\theta_{10}}{\beta} + \frac{\theta_{20}}{\alpha}\right)$ . A simple observation can be made from the above equations that, for large positive value of  $K_1 + K_2$ , the first term (within the integral) in the above equations will tend to zero. Hence,  $x_c \approx x_{c0}$  and  $y_c \approx y_{c0}$ .  $\blacksquare$

We define a parameter  $(K_1/K_2) = \rho$ . For balanced formation, as stated in (13),  $K_1$  and  $K_2$  will satisfy the inequality  $K_1 + K_2 > 0$ , which implies that  $\eta \rho (\rho + 1) > 0$  where  $K_1 = \eta$  and  $K_2 = (\eta/\rho)$ . For  $\omega_0 = 0$ , Fig. 1 shows the locus of the convergence points for different value of  $\eta$  while keeping the ratio  $\rho$  constant. The locus is a straight line, as will be proved in Theorem 3. Here, two convergence points  $(x_{c1}, y_{c1})$  and  $(x_{c2}, y_{c2})$  are shown corresponding to the gains  $\eta_1$  and  $\eta_2$ , respectively. Point  $(x_{c0}, y_{c0})$  is the initial centroid.

**Theorem 3:** For the special two particle case when  $N = 2$  and  $\omega_0 = 0$ , the locus of the convergence points with different  $\eta$  but fixed  $\rho$ , is a straight line converging to the initial centroid for higher values of  $|\eta|$ .

*Proof:* For  $\omega_0 = 0$ , Eqs. (30) and (31) further simplify to the following equations:

$$x_c(t) - x_{c0} = (1/2\lambda) \int_{l'_1}^{l'_2} \sin\left(\delta - \frac{\Omega \phi_0}{2}\right) / \cos\left(\frac{\Omega}{2}\right) d\Omega \quad (32)$$

$$y_c(t) - y_{c0} = (1/2\lambda) \int_{l'_1}^{l'_2} \cos\left(\delta - \frac{\Omega \phi_0}{2}\right) / \cos\left(\frac{\Omega}{2}\right) d\Omega \quad (33)$$

where,  $l'_1 = 2 \arctan((1/\phi))$  and  $l'_2 = 2 \arctan\{(1/\phi e^{-\lambda t})\}$  are, respectively, the lower and upper limits of the integrals. We also note that the integrals in (32) and (33), which can be denoted as  $I_1$  and  $I_2$ , respectively, converge to some constant values when  $t \rightarrow \infty$ . For example,

$$x_c(t \rightarrow \infty) = x_{c0} + h_1 \ln \left| (\phi - 1) + \sqrt{1 + \phi^2} \right| \quad (34)$$

and

$$y_c(t \rightarrow \infty) = y_{c0} + h_2 \ln \left| (\phi - 1) + \sqrt{1 + \phi^2} \right| \quad (35)$$

which are constant (as  $\phi$  is constant) and occur when  $\phi_0 = 0$  (that is,  $K_1 = K_2 = \kappa$ ). Here,  $h_1 = (\cos \delta / \kappa)$  and  $h_2 = (\sin \delta / \kappa)$ . The constants  $\delta$  and  $\phi_0$  depend only on the gain ratio  $\rho$  as  $\delta = \left(\frac{\theta_{10} + \rho \theta_{20}}{\rho + 1}\right)$  and  $\phi_0 = \left(\frac{\rho - 1}{\rho + 1}\right)$ , which ensures that the integrals  $I_1$  and  $I_2$  will individually converge to the same values for different gains  $K_1 (= \eta_1)$  and  $K_2 (= \eta_2/\rho)$  with fixed  $\rho$ . Therefore, for different gains  $\eta_1$  and  $\eta_2$ , (32) and (33) can be obtained for the case when as  $t \rightarrow \infty$  as:

For  $K_i = \eta_i$ ;  $i = 1, 2$

$$\begin{aligned} x_{ci}(t \rightarrow \infty) - x_{c0} &= \rho C_1 / \eta_i (\rho + 1) \\ y_{ci}(t \rightarrow \infty) - y_{c0} &= -\rho C_2 / \eta_i (\rho + 1) \end{aligned}$$

where,  $C_1$  and  $C_2$  are constants. Here, the ratio

$$y_{ci}(t \rightarrow \infty) - y_{c0} / x_{ci}(t \rightarrow \infty) - x_{c0} = -C_2 / C_1, \quad i = 1, 2. \quad (36)$$

implies that the locus of the convergence points is a straight line with slope  $-(C_2/C_1)$ . Also, it is obvious that the higher values of  $|\eta|$  ensures large positive values of  $K_1 + K_2$ . Therefore, the locus of the convergence points approaches the initial centroid  $(x_{c0}, y_{c0})$  for higher values of  $|\eta|$  (from Lemma 1).  $\blacksquare$

**Corollary 2:** In Fig. 1, Let  $d_1$  and  $d_2$  be the respective distances of points  $(x_{c1}, y_{c1})$  and  $(x_{c2}, y_{c2})$  from the initial centroid  $(x_{c0}, y_{c0})$ , then the relation among the parameters  $\eta_1$ ,  $\eta_2$ ,  $d_1$  and  $d_2$ , is given by

$$d_1 \eta_1 = d_2 \eta_2 \quad (37)$$

*Proof:* With reference to Fig. 1, we can write

$$\begin{aligned} d_i &= \sqrt{(x_{ci} - x_{c0})^2 + (y_{ci} - y_{c0})^2} \\ &= \frac{1}{\eta_i} \left( \frac{\rho}{\rho + 1} \right) \sqrt{C_1^2 + C_2^2}, \quad i = 1, 2. \end{aligned}$$

From above equation, we can conclude that  $d_1 \eta_1 = d_2 \eta_2$ .  $\blacksquare$

## V. SYNCHRONIZED FORMATION

The particles are said to be in synchronized formation when the direction of their movement, along with that of their centroid, approaches a common velocity direction  $\theta_c$ . In synchronized formation, the average linear momentum of the group is maximum, that is  $|p_\theta| = 1$ , which occurs when controller gains  $K_k < 0$  for all  $k = 1, \dots, N$ .

After synchronization,

$$\theta_1(t) = \theta_2(t) \dots = \theta_N(t) = \theta_c \quad (38)$$

So, from (23), and (38), we get

$$\theta_c(t) = \omega_0 t + \sum_{i=1}^N (\theta_i(0)/K_i) \left/ \sum_{i=1}^N (1/K_i) \right. \quad (39)$$

Therefore, we can get a desired common direction  $\theta_c$  by an appropriate selection of the gains  $K_k$ .

## VI. SIMULATION RESULTS

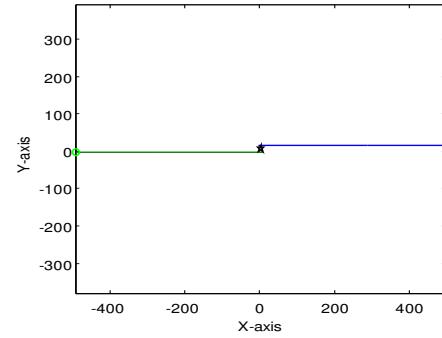
Simulation results are presented for both balanced and synchronized configurations. At first, we discuss the simulation results in the balanced configuration by considering the case of two particles with their initial positions  $Z_0 = [(-2, 15), (6, -5)]$  and initial velocity directions  $D_0 = [30^\circ, 60^\circ]$ , respectively. Therefore, their initial centroid is  $C_0 = (2, 5)$ . The trajectories of the particles are shown in Fig. 2 for the two sets of gains. In Figs. 2(a) and 2(b), the trajectories are shown for the gains  $K' = [K_1', K_2'] = [0.5, 2]$  and  $K'' = [K_1'', K_2''] = [2, -0.4]$  when  $\omega_0 = 0$ , which results in the convergence points  $C'_{fs} = (2.7330, 6.4048)$  and  $C''_{fs} = (4.1392, 5.0780)$  with the corresponding final velocity directions  $D'_{fs} = (0^\circ, 180^\circ)$  and  $D''_{fs} = (-157.5^\circ, 22.5^\circ)$ , respectively. Figs. 2(c) and 2(d) show trajectories for similar gains  $K'$  and  $K''$  when  $\omega_0 = 0.1$ , which results in the convergence points  $C'_{fc} = (2.5624, 6.4406)$  and  $C''_{fc} = (4.2520, 5.4560)$  with the corresponding instantaneous velocity directions  $D'_{fc} = (344.844^\circ, 164.844^\circ)$  and  $D''_{fc} = (187.344^\circ, 7.344^\circ)$ . In all the figures, the symbol ‘★’ represents the final centroid of the particles.

Consider a new pair of gains  $\hat{K} = [\hat{K}_1, \hat{K}_2] = [1, 4]$ , which has the same gain ratio  $\rho = (\hat{K}_1/\hat{K}_2) = (K_1'/K_2') = 0.25$  as  $K'$ . For  $\hat{K}$ , we have the convergence point  $\hat{C}_{fs} = (2.3690, 5.7040)$  when  $\omega_0 = 0$ . With reference to Fig. 1, we get the distances  $d_1 = 1.58$  (which is the distance between  $C'_{fs}$  and  $C_0$ ) when  $\eta_1 = 0.5$  and  $d_2 = 0.79$  (which is the distance between  $\hat{C}_{fs}$  and  $C_0$ ) when  $\eta_2 = 1$ . Hence, we can easily verify that  $d_1\eta_1 = d_2\eta_2 = 0.79$ , as proved in (37).

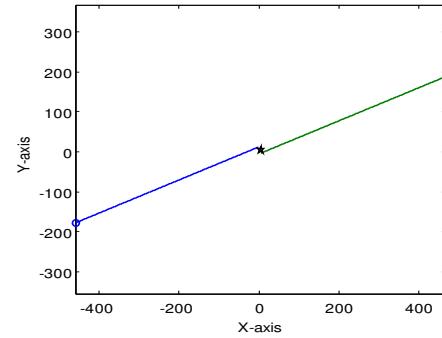
The simulation results in synchronized configuration are shown in Fig. 3, where we consider six particles with their initial positions  $z_0 = [(0, 0), (-2, 15), (-1, 1), (6, -5), (8, 0), (0, 4)]$  and initial velocity directions  $d_0 = [15^\circ, 30^\circ, 45^\circ, 60^\circ, 70^\circ, 120^\circ]$ , respectively. In Figs. 3(a) and 3(b), the trajectories of the particles along with their centroid are, respectively, shown for the gains  $G' = [-1, -3, -5, -7, -9, -11]$  and  $G'' = [-1, -1/3, -1/5, -1/7, -1/9, -1/11]$  when  $\omega_0 = 0$ , which correspondingly result in common directions of motion at an angles  $d'_{fs} = 42.5031^\circ$  and  $d''_{fs} = 69.1667^\circ$ . Similarly, we get the final common directions as  $d'_{fc} = 256.032^\circ$  and  $d''_{fc} = 282.708^\circ$ , corresponding to the gains  $G'$  and  $G''$  when  $\omega_0 = 0.1$ .

## VII. CONCLUSIONS

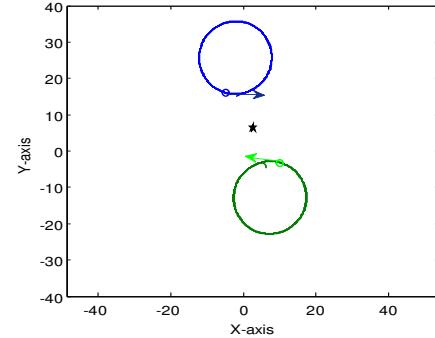
This paper analyses a special kind of cooperative control framework for heterogeneously coupled particles. The proper selection of these heterogenous controller gains has a significant effect on the formation of balanced and synchronized configurations of the particles. In the balanced configuration, we can control both the velocity directions as well as the convergence point while in the synchronized configuration, we can get a desired common direction of motion by selecting heterogeneous gains appropriately. Some interesting results are shown for two particles in balanced configuration (when  $\omega_0 = 0$ ) as the locus of the convergence points for the varying gains is a straight line converging to the initial



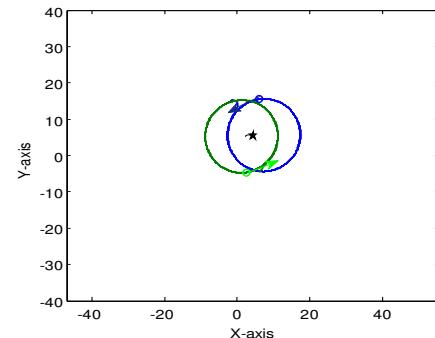
(a) For  $K' = [0.5, 2]$  and  $\omega_0 = 0$



(b) For  $K'' = [2, -0.4]$  and  $\omega_0 = 0$

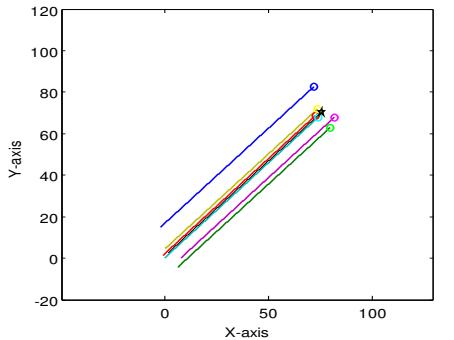


(c) For  $K' = [0.5, 2]$  and  $\omega_0 = 0.1$

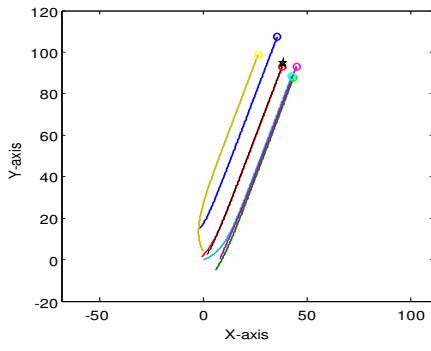


(d) For  $K'' = [2, -0.4]$  and  $\omega_0 = 0.1$

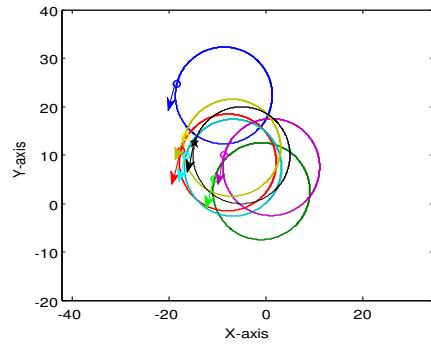
Fig. 2. Balanced trajectories of the two particles along with their convergence point for different gain pairs



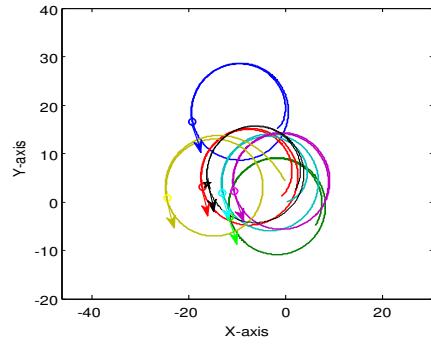
(a) For  $G' = [-1, -3, -5, -7, -9, -11]$  and  $\omega_0 = 0$



(b) For  $G'' = [-1, -\frac{1}{3}, -\frac{1}{5}, -\frac{1}{7}, -\frac{1}{9}, -\frac{1}{11}]$  and  $\omega_0 = 0$



(c) For  $G' = [-1, -3, -5, -7, -9, -11]$  and  $\omega_0 = 0.1$



(d) For  $G'' = [-1, -\frac{1}{3}, -\frac{1}{5}, -\frac{1}{7}, -\frac{1}{9}, -\frac{1}{11}]$  and  $\omega_0 = 0.1$

Fig. 3. Synchronized trajectories of the six particles along with their centroid for different gain pairs

centroid. These properties, related to the heterogenous gains, have a wide range of applications in controlling aerial and underwater vehicles.

### VIII. ACKNOWLEDGEMENTS

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## Stabilization of Balanced Circular Motion About a Desired Center

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**Abstract:** In this paper, we study the collective motion of a group of  $N (\geq 2)$  identical agents trying to achieve a circular formation centered at a desired location, which is fixed. A circular formation is characterized by the motion of all agents around the same circle in the same direction. To solve this problem, we propose a planar motion model that incorporates two control inputs. One of the control inputs is chosen independently and the other control input is decided by using the composite Lyapunov function. We show that the desired location of the center of this circular formation, which is fixed, is obtained by directing the centroid of the group of agents to that desired location. This leads to a collective formation of all the agents, known as balanced circular formation. The theoretical findings are supported by simulations.

**Keywords:** Multiagent system, balanced formation, circular formation.

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### 1. INTRODUCTION

Collective control of multiagent systems has received considerable attention among control engineers to address the challenges related to formation of aerial and underwater vehicles. Depending upon the nature of a wide range of applications such as tracking, surveillance, reconnaissance, search and data collection (Casbeer et al. (2006); Leonard et al. (2007); Paley (2007a)), there is a need for achieving various formations of these vehicles. In this paper, a specific kind of collective formation, in which all the agents of a group move on a common circle in such a way that their centroid (average position of all the agents) and the center of the common circle traversed by them are the same, and remains fixed on a desired location, is considered. This kind of formation of agents is called a balanced circular formation about the desired center (or centroid). The word “balanced” refers to the motion of all the agents such that their movement gives rise to a fixed centroid.

Sepulchre et al. (2007) proposed a steering control law to stabilize synchronized and balanced circular formations of a group of agents. Synchronization refers to the situation where all the agents, along with that of their centroid, move in the same direction. These formations, as described in Sepulchre et al. (2007), are shown to be stabilized to a common circle centered at an arbitrary and desired locations. Contrary to their analysis, in this paper, by using a modified planar motion model of the agent, the possibility of getting a desired center, only in the case of balanced circular formation, is explored. In our earlier work Jain et al. (2014) (unpublished), the stabilization of collective formation of a group of  $N$ -agents, where each agent either moves in a straight line or rotates on

individual circle, in a manner that their centroid remains fixed at the desired location, is discussed. As an extension, in this paper, the control laws are proposed to get a balanced circular formation, where all the agents orbit a common circle centered at a desired location, which is fixed.

Recently, a various control strategies are proposed to obtain different structures of circular formations of a group of autonomous agents. Paley et al. (2005) derived a control law to stabilize the splay state formation in which the agents spread out equidistantly on a circle. Xu et al. (2013) provided similar results by utilizing a modified Kuramoto model (Strogatz (2000)). Paley (2008) proposed a Lyapunov-based design methodology to stabilize the collective circular motion of the agents in a known uniform and constant flow field. A further extension to this problem is given by Paley and Peterson (2009), where the flow-field is time-invariant but assumed to vary in space. To attain the parallel and circular formations of the agents, the backstepping approach of designing the control algorithms is discussed by Mellish and Paley (2010). An interesting related work on achieving collective circular motion is given in Chen and Zhang (2011, 2013), where the authors consider a heterogeneous model for the agents in the sense of their linear and rotational speeds.

The main contribution of this paper is to propose a strategy to stabilize the balanced circular formation of a group of identical multiagent system about a desired location of the center. We show that the agents, which start from arbitrary initial positions and with arbitrary initial velocity directions, stabilize to a circular formation in such a way that the center of the common circle and the centroid of the group, both converge to a common

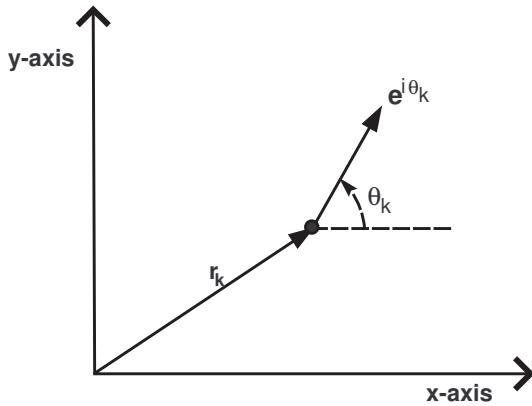


Fig. 1. Orientation of the  $k$ -th agent in the 2D plane for dynamics (1a) and (1b).

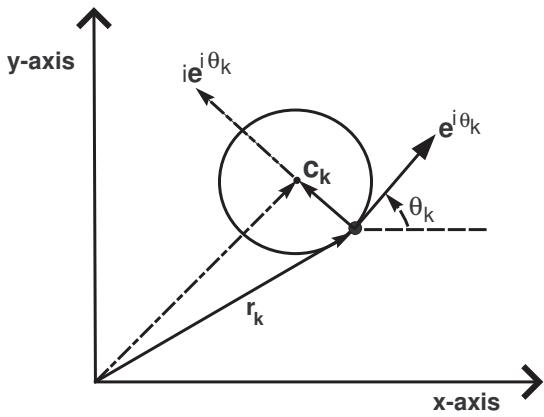


Fig. 2.  $k$ -th agent rotating on a circle in the 2D plane

desired point. For this purpose, we use the tools from the Lyapunov theory and the LaSalle's Invariance principle (Khalil (2000)) to prove the stability of the collective motion of  $N$ -agents.

The rest of the paper is organized as follows. In Section II, we state a few definitions and describe the system dynamics. In Section III, the feedback control laws are proposed to achieve the desired formation of the agents. In Section IV, Lyapunov based approach is provided to assure the stability of the desired collective formation. The simulation results are discussed in Section V.

## 2. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Previous work in this area has focused on the collective dynamics of  $N$  autonomous agents in which each agent (assumed to have unit mass) moves at unit speed in a planar space. The equations of the motion of the  $k$ -th agent are (Paley et al. (2007b)):

$$\dot{r}_k = e^{i\theta_k} \quad (1a)$$

$$\dot{\theta}_k = u_k, \quad k = 1, \dots, N \quad (1b)$$

Here, the position of the  $k$ -th agent is  $r_k$ , while its velocity is  $\dot{r}_k = e^{i\theta_k}$ , where  $i = \sqrt{-1}$  is the standard complex number and  $\theta_k$  denotes the orientation of the velocity

vector from the positive x-axis. In Paley et al. (2007b), the direction,  $\theta_k$  of the velocity vector of  $k$ -th agent is also referred to as its phase. The turn rate,  $\dot{\theta}_k$  of the  $k$ -th agent is determined by the steering control law  $u_k$ . The interpretation of (1a) and (1b) is given in Fig. 1.

It is worth noting that, if, for all  $k = 1 \dots N$ , the control input  $u_k$  is identically zero, then each agent travels in a straight line in its initial direction  $\theta_k(0)$ . If the control input  $u_k = \omega_0$  is constant and non zero, then each agent rotates on a circle with radius  $\rho_0 = |\omega_0|^{-1}$ . This situation is shown in Fig. 2, where the center of the circle traversed by the  $k$ -th agent is  $c_k$ . The multiplication of the complex number  $i$  to the velocity vector  $e^{i\theta_k}$  provides a vector  $ie^{i\theta_k}$ , which is in the direction perpendicular to the velocity vector  $e^{i\theta_k}$  and points towards the center  $c_k$  (as shown in Fig. 2). Therefore, by using the law of vector addition, the center of the circle traversed by the  $k$ -th agent is,

$$c_k = r_k + i\omega_0^{-1}e^{i\theta_k} \quad (2)$$

Now, we state the following definitions from the previous literature (Sepulchre et al. (2007)).

**Definition 1:** Let  $c_k$  be given by (2),  $\mathbf{c} \triangleq (c_1, \dots, c_N)^T \in \mathbb{C}^N$ ,  $\boldsymbol{\theta} \triangleq (\theta_1, \dots, \theta_N)^T$ ,  $\mathbf{1} \triangleq (1, \dots, 1)^T \in \mathbb{R}^N$ ,  $c_0 \in \mathbb{C}$ ,  $\omega_0 \in \mathbb{R}$ , and  $\omega_0 \neq 0$ . A circular formation of the agent's model defined in (1a) and (1b) is the set of trajectories for which  $\dot{\boldsymbol{\theta}} = \omega_0 \mathbf{1}$  and  $\mathbf{c} = c_0 \mathbf{1}$ , that is, all the agents travel around a circle of radius  $|\omega_0|^{-1}$  and the centers of all the circles coincide to a common point  $c_0$ . The direction of rotation is determined by the sign of  $\omega_0$ . If  $\omega_0 > 0$ , then all the agents rotate in the counterclockwise direction. Whereas, if  $\omega_0 < 0$ , then all the agents rotate in the clockwise direction. The relative phases of the agents in a circular formation are arbitrary.

**Definition 2:** A balanced circular formation of the agent's model defined in (1a) and (1b) is a circular formation of all the agents such that their motion along the common circle possesses a fixed centroid  $c_0$ , that is, the locations of the center of the common circle and the centroid of the group coincide at a point  $c_0$  and remain stationary.

In order to reach the objective of this paper, a modified agent's model is proposed as (Jain et al. (2014))

$$\dot{r}_k = e^{i\theta_k} + u_0 = v_k e^{i\psi_k} \quad (3a)$$

$$\dot{\theta}_k = u_k, \quad k = 1, \dots, N \quad (3b)$$

Here,  $u_0 \in \mathbb{C}$  is an explicitly appended control input, which assumes the same values for all the agents and controls both the magnitude and the direction of the resultant velocity vector of each agent. Later we discuss in the next section how to choose  $u_0$  to get a desired behavior of the velocity vector of each agent. The velocity of the  $k$ -th agent is  $v_k \triangleq |e^{i\theta_k} + u_0|$ , while the heading angle of the  $k$ -th agent is  $\psi_k$ . The other parameters are similarly defined as in (1a) and (1b). The interpretation of (3a) and (3b) is shown in Fig.3.

## 3. DESIGN OF CONTROL LAWS

The problem addressed in this paper is solved in three steps. At first, a Lyapunov based potential function is

described whose minimization leads to the circular formation. Later, the convergence of the current centroid of the group to a desired location  $c_0$  is shown by minimizing another potential function which incorporates the selection of the control input  $u_0$ . Finally, the control law  $u_k$  is proposed by minimizing the composite Lyapunov function consisting of these potential functions.

### 3.1 Achieving Circular Formation

A circular formation is obtained when all the centers of the circles traversed by each agent coincide. This corresponds to the following algebraic condition (Sepulchre et al. (2007))

$$P\mathbf{c} = 0; \quad P = I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^T. \quad (4)$$

Here,  $P$  is a symmetric projection matrix and satisfies  $P = P^T$  and  $P^2 = P$ . The matrix  $P$  has only two distinct eigenvalues: zero, which has multiplicity one, and one, which has multiplicity  $N - 1$ . Therefore,  $P\mathbf{c} = 0$  if and only if  $\mathbf{c} = c_0\mathbf{1}$ . For ease of calculation, we multiply  $c_k$  by a constant factor  $-i\omega_0$  to get a new variable

$$s_k = -i\omega_0 c_k = e^{i\theta_k} - i\omega_0 r_k. \quad (5)$$

Therefore, in order to achieve the circular formation the condition (4) can be equivalently stated as

$$P\mathbf{s} = 0; \quad P = I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^T. \quad (6)$$

This suggest the minimization of the following candidate Lyapunov function (Sepulchre et al. (2007))

$$S(\mathbf{r}, \boldsymbol{\theta}) = \frac{1}{2} \|P\mathbf{s}\|^2 = \frac{1}{2} \langle P\mathbf{s}, P\mathbf{s} \rangle^2 \quad (7)$$

Here,  $\langle z_1, z_2 \rangle = \text{Re}\{\bar{z}_1 z_2\}$ , represents the inner product of complex numbers  $z_1, z_2 \in \mathbb{C}$ , where  $\bar{z}_1$  denotes the complex conjugate of  $z_1$ . Analogously, for vectors, the boldface notation  $\langle \mathbf{z}_1, \mathbf{z}_2 \rangle = \text{Re}\{\bar{\mathbf{z}}_1 \mathbf{z}_2\}$ , is used. The bold-face parameters  $\mathbf{s}$  and  $\mathbf{r}$ , respectively, denote the  $N$ -vector  $\mathbf{s} = (s_1, \dots, s_N)^T$  and  $\mathbf{r} = (r_1, \dots, r_N)^T$ . Note that the potential  $S(\mathbf{r}, \boldsymbol{\theta})$  is a positive semi-definite function and approaches zero when  $P\mathbf{s} = 0$ .

By differentiating (5) along the solutions of (3a) and (3b), we get

$$\dot{s}_k = ie^{i\theta_k}(u_k - \omega_0) - i\omega_0 u_0. \quad (8)$$

Taking the time derivative of the Lyapunov function (7) along the trajectories of (3a) and (3b) yields

$$\dot{S} = \langle P\mathbf{s}, P\dot{\mathbf{s}} \rangle = \sum_{k=1}^N \langle P_k \mathbf{s}, ie^{i\theta_k} \rangle (u_k - \omega_0) \quad (9)$$

where,  $P_k$  denotes the  $k$ -th row of the matrix  $P$ . Since

$$P_k \mathbf{s} = s_k - \frac{1}{N} \mathbf{1}^T \mathbf{s} = e^{i\theta_k} - i\omega_0 r_k - \left( \frac{1}{N} \sum_{j=1}^N e^{i\theta_j} - i\omega_0 R \right) \quad (10)$$

we obtain

$$\begin{aligned} \langle P_k \mathbf{s}, ie^{i\theta_k} \rangle &= -\langle \omega_0(r_k - R), e^{i\theta_k} \rangle - \left\langle \frac{1}{N} \sum_{j=1}^N e^{i\theta_j}, ie^{i\theta_k} \right\rangle \\ &= -\langle \omega_0 \hat{r}_k, e^{i\theta_k} \rangle - \langle p_\theta, ie^{i\theta_k} \rangle \end{aligned} \quad (11)$$

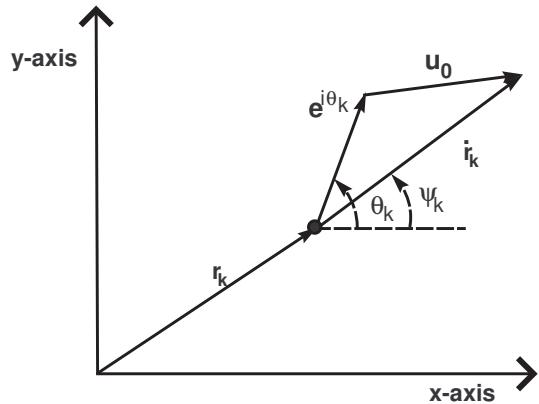


Fig. 3. Orientation of the  $k$ -th agent in the 2D plane for dynamics (3a) and (3b).

where  $R = \frac{1}{N} \sum_{k=1}^N r_k$ ,  $\hat{r}_k = r_k - R$  and  $p_\theta = \frac{1}{N} \sum_{j=1}^N e^{i\theta_j}$ . Using (11), (9) can be rewritten as

$$\dot{S} = - \sum_{k=1}^N (\omega_0 \langle \hat{r}_k, e^{i\theta_k} \rangle + \langle p_\theta, ie^{i\theta_k} \rangle) (u_k - \omega_0) \quad (12)$$

This expression will be used in sequel for further simplifications.

### 3.2 Convergence of Group Centroid to the Desired Location

Let the locations of the current and the desired centroid of a group of  $N$  agents be  $r_c$  and  $r_{cd} (= c_0)$ , respectively. Note that the location of the desired centroid  $r_{cd}$  is selected similar to the center  $c_0$  of the common circle. It will be proved later that, after stabilization to balanced circular formation, the center  $c_0$  of the common circle is the centroid of the group. This was the motivation behind the idea of achieving convergence of the centroid of the group to a desired location  $c_0$ , which is the center of the common circle. From (3a), the rate of change of the position of the current centroid  $\dot{r}_c$  is,

$$\dot{r}_c = \frac{1}{N} \sum_{k=1}^N e^{i\theta_k} + u_0 \quad (13)$$

To ensure consensus between the positions of the current and the desired centroids, the concept of the reference velocity, which is the commanded velocity of the centroid of the group of agents, is adapted from Klein and Morgansen (2006). The reference velocity  $\dot{r}_{ref}$  is defined as

$$\dot{r}_{ref} = f(D)\hat{u}. \quad (14)$$

where,  $f(D)$  is a smooth function of the variable  $D = \|r_{cd} - r_c\|$ , which is the distance between the current and the desired centroid. The parameter  $\hat{u} = \frac{r_{cd} - r_c}{\|r_{cd} - r_c\|}$ , is a unit vector along the straight line joining the current and the desired centroids. The reference velocity is defined in such a way that it must satisfy the condition  $\lim_{D \rightarrow 0} f(D) = 0$ , when the locations of the current and the desired centroids coincide (Jain et al. (2014)).

Now, let the error between the velocity of the collective centroid,  $\dot{r}_c$ , and reference velocity,  $\dot{r}_{ref}$ , be

$$e_v = \dot{r}_c - \dot{r}_{ref} \quad (15)$$

Consider a candidate Lyapunov function (Jain et al. (2014)):

$$U(\boldsymbol{\theta}) = \frac{N}{2} \|e_v\|^2 = \frac{N}{2} \|\dot{r}_c - \dot{r}_{ref}\|^2 \quad (16)$$

In (13), let  $u_0$  be as

$$u_0 = \dot{r}_{ref} \quad (17)$$

to have

$$\dot{r}_c - \dot{r}_{ref} = \frac{1}{N} \sum_{k=1}^N e^{i\theta_k} = p_\theta \quad (18)$$

and hence

$$U(\boldsymbol{\theta}) = \frac{N}{2} \|p_\theta\|^2 \quad (19)$$

The time derivative of the Lyapunov function  $U$ , along the trajectories (3a) and (3b), is

$$\dot{U} = N \sum_{k=1}^N \left\langle p_\theta, \frac{\partial p_\theta}{\partial \theta_k} u_k \right\rangle \quad (20)$$

Using (18), we get

$$\dot{U} = \sum_{k=1}^N \left\langle p_\theta, ie^{i\theta_k} \right\rangle u_k \quad (21)$$

The above expression can also be rewritten as

$$\dot{U} = \sum_{k=1}^N \left\langle p_\theta, ie^{i\theta_k} \right\rangle (u_k - \omega_0) \quad (22)$$

It is because of the property  $\langle \Delta, \mathbf{1} \rangle = 0$ , where  $\Delta_k = \langle p_\theta, ie^{i\theta_k} \rangle$  and  $\Delta = (\Delta_1, \dots, \Delta_N)^T$ .

### 3.3 Composite Lyapunov Function and Control Law $u_k$

Consider a composite Lyapunov function

$$V(\mathbf{r}, \boldsymbol{\theta}) = S(\mathbf{r}, \boldsymbol{\theta}) + U(\boldsymbol{\theta}). \quad (23)$$

The time derivative of the Lyapunov function  $V$ , along the trajectories (3a) and (3b), is

$$\dot{V} = \dot{S} + \dot{U} \quad (24)$$

Using (12) and (22) into (24) yields

$$\dot{V} = -\omega_0 \sum_{k=1}^N \left\langle \hat{r}_k, e^{i\theta_k} \right\rangle (u_k - \omega_0) \quad (25)$$

Choosing the control law

$$u_k = \omega_0 (1 + K \langle \hat{r}_k, e^{i\theta_k} \rangle) \quad (26)$$

results in

$$\dot{V} = -K \omega_0^2 \sum_{k=1}^N \langle \hat{r}_k, e^{i\theta_k} \rangle^2. \quad (27)$$

which is negative semi-definite for all  $K > 0$ . By LaSalle's Invariance principle (Khalil (2000)) all solutions of the system defined in (3a) and (3b), asymptotically converge to the largest invariant set  $M$  where

$$\langle \hat{r}_k, e^{i\theta_k} \rangle = 0 \quad (28)$$

for all  $k = 1, \dots, N$ . In this set, the dynamics reduce to  $\dot{\theta}_k = \omega_0$ , which implies that  $\dot{\mathbf{s}} = 0$  (as the asymptotic value of  $u_0 = 0$ ). It means that  $P\mathbf{s} = 0$  and hence  $P\mathbf{c} = 0$ . As a result,  $\mathbf{c} = c_0\mathbf{1}$  for some fixed  $c_0 \in \mathbb{C}$ , that is, the centers of the individual circles traversed by all the agents

coincide. Moreover, due to the non-increasing nature of  $V$ , the potential  $U$  also approaches zero. Hence, the solutions converge to a circular relative equilibrium in the critical set of  $U$  where the asymptotic value of  $p_\theta = 0$ .

## 4. STABILITY ANALYSIS

In this section, it is shown that the locations of the current and the desired centroid also coincide when the velocity of the current centroid matches with the reference velocity. Therefore, the current centroid of the agents converges to the desired point  $c_0$ .

Consider the Lyapunov function

$$\tilde{V}(\mathbf{r}) = \frac{1}{2} \|r_{cd} - r_c\|^2. \quad (29)$$

The time derivative of the Lyapunov function,  $V(\mathbf{r})$ , is given as

$$\dot{\tilde{V}}(\mathbf{r}) = \langle r_{cd} - r_c, \dot{r}_{cd} - \dot{r}_c \rangle \quad (30)$$

Using  $\dot{r}_c = \dot{r}_{ref}$  and (14), gives

$$\begin{aligned} \dot{\tilde{V}}(\mathbf{r}) &= \left\langle r_{cd} - r_c, -f(D) \frac{r_{cd} - r_c}{\|r_{cd} - r_c\|} \right\rangle \\ &= -Df(D) \end{aligned} \quad (31)$$

Following the Lyapunov stability requirements (Khalil (2000)), it can be concluded that the condition  $f(D) > 0$ ,  $\forall D$ , implies that the potential  $\tilde{V}(\mathbf{r})$  is a decreasing function and approaches zero when  $r_c = r_{cd} = c_0$ , as desired. Therefore, by choosing the control laws  $u_0$  and  $u_k$ , respectively, as in (17) and (26), balanced circular formation at the desired center  $c_0$ , is stabilized.

After stabilized to balanced circular formation centered at  $c_0$ , the centers of individual circles traversed by each agent satisfies

$$c_1 = c_2 = \dots = c_N = c_0. \quad (32)$$

By taking the average of (2) on both the sides and using the fact that, after stabilization to balanced circular formation,  $p_\theta = 0$ . We have

$$c_0 = \frac{1}{N} \sum_{k=1}^N r_k \quad (33)$$

Therefore, after stabilization to balanced circular formation, the center of the common circle is the centroid of the group of agents.

## 5. SIMULATION RESULTS

Simulations results are provided for  $N = 20$  agents with random initial positions and initial headings. Fig. 4 shows the the trajectories of the agents along with their centroid (shown by dashed line) for  $f(D) = 2(1 - e^{-0.5D})$ , angular frequency  $\omega_0 = 0.1$  and the controller gain  $K = 0.1$ . The agents converge to a common circle, which is centered at the desired location  $(10, 10)$  and has the radius  $\rho_0 = |\omega_0|^{-1} = 10$  units. Fig. 5 depicts the variation of function  $f(D)$  with time, which approaches zero with time. The variation of the argument  $\theta_D$  of the control input  $u_0$  with time, is shown in Fig. 6, which shows that the agents go round the centroid as they converge to final desired circle.

For better visualization of the paths traversed by the agents, the trajectory of one of the agents (say the  $k$ -th

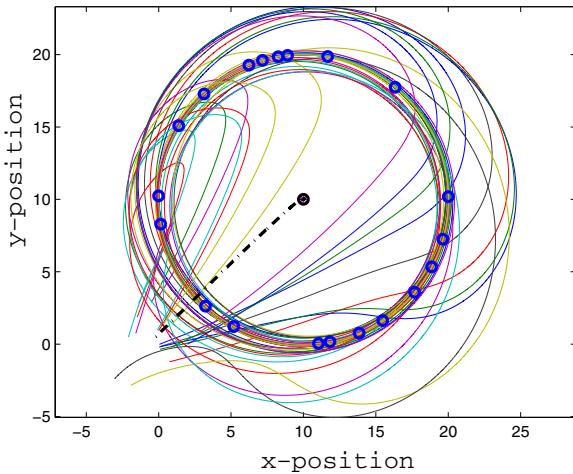


Fig. 4. Balanced circular formation of  $N = 20$ , agents in the 2D plane for  $f(D) = 2(1 - e^{-0.5D})$ , and  $\omega_0 = K = 0.1$ . The desired location of the center is at  $(10, 10)$ .

agent,  $k = 1$ ) is shown in Fig. 7, where it converges to the desired circle. Fig. 8 represents the variation of the control input  $u_k$  of the  $k$ -th agent with time, which approaches to a constant value  $u_{kf} = 0.1$ . It happens because of the fact that, after stabilization, the control input  $u_k$  must satisfy  $u_{kf} = \omega_0 = 0.1$ , for all  $k = 1, \dots, N$ . Therefore, the simulation results are in accordance with the theoretical findings.

Note that, in Fig. 4, the agents are rotating about the desired center such that their relative positions on the common circle are arbitrary. To get a balanced circular formation where the agents are equally spaced on the common circle, is an interesting future work.

According to the need of various practical applications, it would be interesting to extend the problem addressed in this paper to account for bounded variations in the velocity of agents with time. The results of this paper can also be further generalized to stabilize balanced formations around ellipsoidal trajectories.

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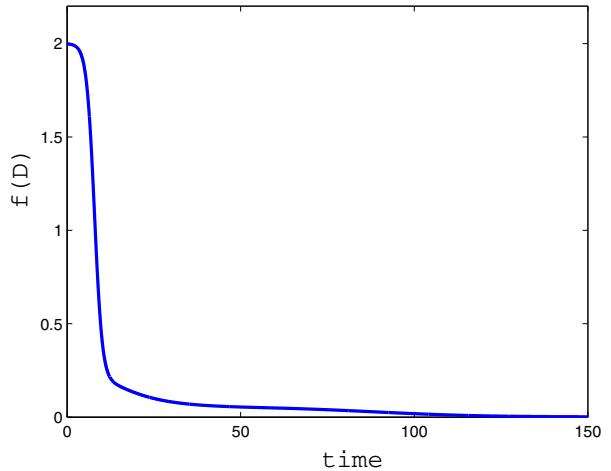


Fig. 5. Variation of  $f(D)$  with time, where  $f(D)$  is the magnitude of the control input  $u_0 = f(D)e^{i\theta_D}$ .

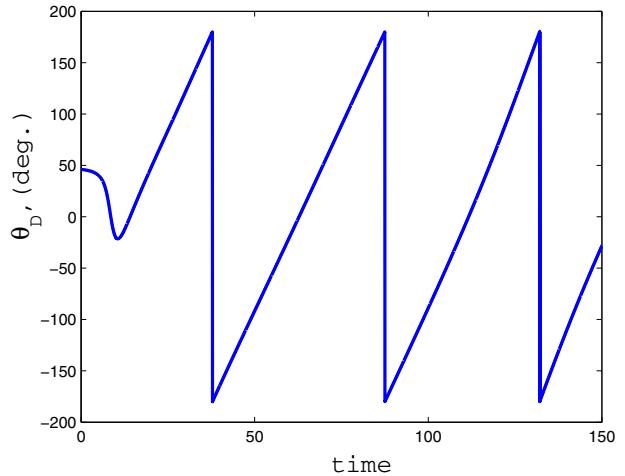


Fig. 6. Variation of  $\theta_D$  with time, where  $\theta_D$  is the argument of control input  $u_0 = f(D)e^{i\theta_D}$ .

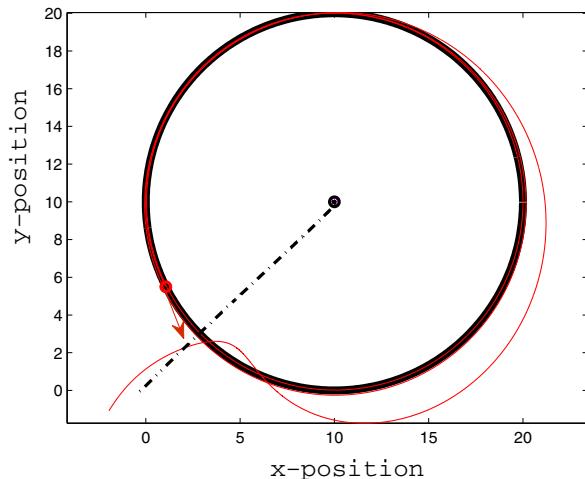


Fig. 7. Trajectory of one of the agents (say the  $k$ -th agent) in 2D plane

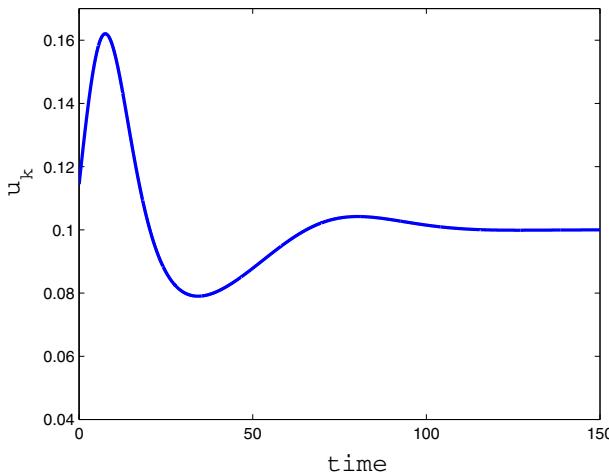


Fig. 8. Variation of the control input  $u_k$  of the  $k$ -th agent with time.

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# Stabilization of Collective Motion in Synchronized, Balanced and Splay Phase Arrangements on a Desired Circle

Anoop Jain<sup>1</sup> and Debasish Ghose<sup>2</sup>

**Abstract**—This paper proposes a design methodology to stabilize collective circular motion of a group of  $N$ -identical agents moving at unit speed around individual circles of different radii and different centers. The collective circular motion studied in this paper is characterized by the clockwise rotation of all agents around a common circle of desired radius as well as center, which is fixed. Our interest is to achieve those collective circular motions in which the phases of the agents are arranged either in synchronized, in balanced or in splay formation. In synchronized formation, the agents and their centroid move in a common direction while in balanced formation, the movement of the agents ensures a fixed location of the centroid. The splay state is a special case of balanced formation, in which the phases are separated by multiples of  $2\pi/N$ . We derive the feedback controls and prove the asymptotic stability of the desired collective circular motion by using Lyapunov theory and the LaSalle's Invariance principle.

**Keywords:** Multiagent system; synchronization; balancing; splay state; desired common circle.

## I. INTRODUCTION

Depending upon the nature of a wide range of engineering applications such as tracking, surveillance, reconnaissance, environmental monitoring, searching, sensing and data collection, various collective motions of the multiagent system have been explored in the recent years. The particularly interesting collective motion considered in this paper is the collective circular motion, in which the agents move on a common circle of desired radius and center in a way such that their headings are either in synchronized, in balanced, or in splay formation. Synchronization refers to the situation when, at all times, the agents, along with that of their centroid, which is the average position of the group of agents, have a common velocity direction. A complementary notion of synchronization is balancing, in which the headings of the agents on the desired common circle, are arranged in such a way that their centroid remains fixed. The splay state formation, which is a special case of the balanced formation, refers to the situation in which the headings of the agents on the desired common circle, are separated by a multiple of  $2\pi/N$ , where  $N$  is the number of agents. Note that the phrase “the desired common circle” is an abbreviation of the phrase “the common circle of desired radius and center, which is fixed”.

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The present work is inspired by the problem addressed in [1] and [2], where the control laws are proposed to stabilize the collective motion of a group of agents about a common circle. In [1], the splay formation of a group of agents is stabilized on a common circle of prescribed radius, which is centered at the origin. On the other hand, in [2], the synchronized and balanced collective motions of the agents, initially rotating around individual circles of same radius, are stabilized on a common circle, which is centered at the prescribed location and have a radius similar to that of initial individual circles. However, unlike previous work in [1] and [2], in this paper, control laws are proposed to stabilize the collective motion on a desired common circle by assuming that the agents are initially rotating around individual circles of different radii. In a similar context, Napora and Paley in [3] describe an observer-based feedback control algorithm to stabilize circular formation using measurements of the relative position only. However, in the present work, control scheme is proposed not only to stabilize the collective motion of agents around a desired common circle but also to achieve synchronized, balanced and splay formations of their phase angles.

There exists an ample literature related to the study of collective circular motion of multiagent system. In [4], control laws are proposed to stabilize the collective circular motion of nonholonomic vehicles around a virtual reference beacon, which is either stationary or moving. Similar results are given in [5], where a Lyapunov guidance vector field approach is used to guide a team of unmanned aircraft to fly a circular orbit around a moving target with prescribed inter-vehicle angular spacing. In [6], Chen and Zhang propose a decentralized control algorithm for a group of nonholonomic vehicles to form a class of collective circular motion, in which the vehicles are evenly distributed over the motion circle, and have the same rotational radius. The latter assumption is relaxed in [7], where the agents move in circles around a common center, but with different radii. Arranz et al. in [8], [9] provide the control algorithms to stabilize the collective motion of multiagent systems around a circular orbit, which has either a fixed radius and time-varying center [8], or a fixed center and time-varying radius [9]. The stabilization of collective circular motion in a uniform and constant flow-field, is given in [10], [11]. These results are further extended in [12] for the time-varying flow-field. In [13], the splay circular formation of multiple robots is stabilized by using a modified Kuramoto model. In [14], the circular motion of agents in the symmetric patterns of their phases, are investigated for a ring-like coupled network. A popular

collective circular motion of multivehicle system under cyclic pursuit is given in [15].

The main contribution of this paper is to propose a control strategy to stabilize the collective motion of a group of agents on a desired common circle with their phase arrangements either in synchronized, in balanced or in splay formation, while allowing the radii and centers of individual, initial circular motions, performed by the agents, to be different. With this purpose, in this paper, we consider the model of identical, all-to-all coupled agents moving in a plane at constant, unit speed and propose a control scheme which incorporates two feedback controllers. The dynamics of each agent is represented by a state vector, which includes the position, heading and angular frequency of each agent as its elements.

The paper is organized as follows. In Section II, we describe the dynamical model of the system and formulate the problem. In Section III, we propose feedback control laws to stabilize collective motion of agents in synchronized and balanced formations on a desired common circle. The control laws to stabilize splay formation of agents on a desired common circle are proposed in Section IV. Simulation results are discussed in Section V. The concluding remarks appear in Section VI.

## II. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Consider a group of  $N$  identical agents in which each agent (assumed to have unit mass) moves at unit speed on a plane. We identify plane  $\mathbb{R}^2$  with the complex plane  $\mathbb{C}$  and use complex variables to describe the position and velocity of each agent. For  $k = 1, \dots, N$ , the position of the  $k$ -th agent is  $r_k \in \mathbb{C}$ , while the velocity of  $k$ -th agent is  $\dot{r}_k = e^{i\theta_k} \in \mathbb{C}$ , where,  $\theta_k \in S^1$  is the orientation of the (unit) velocity vector of the  $k$ -th agent from the real axis, and  $i = \sqrt{-1}$  denotes the standard complex number. The orientation,  $\theta_k$  of the velocity vector is also referred to as the phase of the  $k$ -th agent [16], [17]. Let  $\omega_k \in \mathbb{R}$  be the angular frequency of the circular orbit performed by the  $k$ -th agent. With these notations, the dynamical equations of the motion for the  $k$ -th agent are

$$\dot{r}_k = e^{i\theta_k} \quad (1a)$$

$$\dot{\theta}_k = -\omega_k + u_k \quad (1b)$$

$$\dot{\omega}_k = \mu_k, \quad k = 1, \dots, N \quad (1c)$$

where,  $u_k \in \mathbb{R}$  and  $\mu_k \in \mathbb{R}$  are the feedback control laws, which control the heading and angular frequency of the  $k$ -th agent, respectively.

If, for all  $k = 1, \dots, N$ , the control laws  $u_k$  and  $\mu_k$  are identically zero, then each agent travels at constant, unit speed on a circle of radius  $\rho_k = |\omega_k|^{-1}$ . The convention of direction of rotation on the circle followed in this paper is, if, for all  $k = 1, \dots, N$ , the angular frequency  $\omega_k > 0$ , then all the agents rotate in the clockwise direction and if, for all  $k = 1, \dots, N$ , the angular frequency  $\omega_k < 0$ , then all the agents rotate in the anticlockwise direction. This convention is based on the observation that under the controls  $u_k = \mu_k = 0$  for all  $k = 1, \dots, N$ , the rate of change of heading of the  $k$ -th agent  $\dot{\theta}_k$  is positive when  $\omega_k < 0$ , and it is negative when  $\omega_k > 0$ .

Let the initial motion of the agents with dynamics (1a), (1b) and (1c) be governed by the open-loop controls  $u_k = \mu_k = 0$ , for all  $k = 1, \dots, N$ . In this situation, the  $k$ -th agent moves in a circular orbit with angular frequency  $\omega_k$  about an arbitrary fixed center. We seek to find the feedback controls  $u_k$  and  $\mu_k$  such that the collective motion of the agents is stabilized to a common circle of desired radius  $\rho_0 = \Omega_0^{-1} > 0$  and desired center  $c_0$  (which is fixed) with their phases, either in synchronized, in balanced or in splay formations. Previous work in this direction [2] has focused on achieving the common circular motion about a desired center when the angular frequencies of the initial circular motions of the agents, are same. As an extension, in this paper, we consider that the angular frequencies of the initial circular motions of the agents are different. Note that the angular frequency of the desired circular motion is  $\Omega_0 > 0$ , therefore, in the equilibrium, all the agents, initially rotating in either clockwise or anticlockwise directions, move in the clockwise direction on the desired circle. Note that, this paper does not deal with issue of collision avoidance among agents.

We introduce a few more notations and state a few important results, which are used further in this paper. We use the bold face letters  $\mathbf{r} = (r_1, \dots, r_N)^T \in \mathbb{C}^N$ ,  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_N)^T \in \mathbb{T}^N$ , where  $\mathbb{T}^N$  is the  $N$ -torus, which is equal to  $S^1 \times \dots \times S^1$  ( $N$ -times), and  $\boldsymbol{\omega} = (\omega_1, \dots, \omega_N)^T \in \mathbb{R}^N$  to represent the vectors of length  $N$  for the agent's positions, headings and angular frequencies, respectively. Next, we define the inner product  $\langle z_1, z_2 \rangle$  of the two complex numbers  $z_1, z_2 \in \mathbb{C}$  as  $\langle z_1, z_2 \rangle = \text{Re}(\bar{z}_1 z_2)$ , where  $\bar{z}_1$  represents the complex conjugate of  $z_1$ . Some of the important properties of the inner product  $\langle \cdot, \cdot \rangle$ , which are relevant in the framework of the present paper can be found in [19], and are listed below:

- P1)  $\langle z_1, z_1 \rangle = |z_1|^2$
- P2)  $\langle z_1, z_2 \rangle = \langle z_2, z_1 \rangle$
- P3)  $\langle iz_1, z_2 \rangle = -\langle z_1, iz_2 \rangle$
- P4)  $\langle z_1, cz_2 \rangle = c \langle z_1, z_2 \rangle; \quad c \in \mathbb{R}$
- P5)  $\langle z_1 + w, z_2 \rangle = \langle z_1, z_2 \rangle + \langle w, z_2 \rangle; \quad w \in \mathbb{C}$
- P6)  $\frac{d}{dt} \langle z_1, z_1 \rangle = 2 \langle z_1, \dot{z}_1 \rangle$

These properties will be used in the next sections to simplify the algebraic relations.

## III. DESIGN OF CONTROL LAWS

The problem, addressed in this paper, is solved in four steps. At first, a spacing potential is proposed, the minimization of which leads to the circular formation where all the agents orbit the same point. Then, a phase potential is described, the minimization of which corresponds to the balanced formation, and the maximization corresponds to the synchronized formation. After it, a potential function whose minimization leads to the desired angular frequency of all the agents around a common circle, is suggested. Finally, the control laws  $u_k$  and  $\mu_k$  are proposed by minimizing

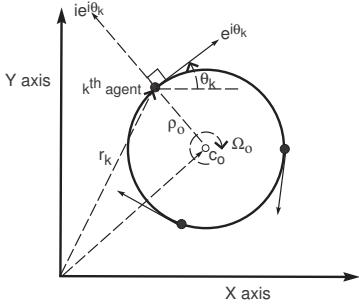


Fig. 1. Orientation of the  $k$ -th agent on the desired circle

a composite Lyapunov function, which combines all these potentials.

#### A. Achieving Desired Circular Formation

The circular formation of all the agents corresponds to their collective motion on a common circle in the same direction of rotation. This situation is shown in figure 1, where each agent rotates at a constant, unit speed in the clockwise direction on a circle of radius  $\rho_0$  and center  $c_0$ . The multiplication of the complex number  $i$  to the velocity vector  $e^{i\theta_k}$  provides a unit vector  $ie^{i\theta_k}$ , which is perpendicular to the velocity vector  $e^{i\theta_k}$  and points away from the center  $c_0$ . From figure 1, the position of the  $k$ -th agent on this circle is given by

$$r_k = c_0 + i\rho_0 e^{i\theta_k} \quad (2)$$

To stabilize the clockwise circular motion with desired angular frequency  $\Omega_0 = \rho_0^{-1} > 0$  about a desired center  $c_0$ , which is fixed, we introduce an error variable  $e_k = r_k - (c_0 + i\rho_0 e^{i\theta_k})$  for all  $k = 1, \dots, N$  and choose a candidate Lyapunov function

$$\begin{aligned} S(\mathbf{r}, \boldsymbol{\theta}) &= \frac{1}{2} \sum_{k=1}^N |e_k|^2 = \frac{1}{2} \sum_{k=1}^N |r_k - c_0 - i\rho_0 e^{i\theta_k}|^2 \\ &= \frac{1}{2} \sum_{k=1}^N \langle r_k - c_0 - i\rho_0 e^{i\theta_k}, r_k - c_0 - i\rho_0 e^{i\theta_k} \rangle \end{aligned} \quad (3)$$

which is minimized when  $r_k = c_0 + i\rho_0 e^{i\theta_k}$  for all  $k = 1, \dots, N$ .

Taking the time derivative of the Lyapunov function (3) along the dynamics (1a), (1b) and (1c), yields

$$\dot{S}(\mathbf{r}, \boldsymbol{\theta}) = \sum_{k=1}^N \langle r_k - c_0 - i\rho_0 e^{i\theta_k}, e^{i\theta_k} (1 - \rho_0(\omega_k - u_k)) \rangle \quad (4)$$

Using properties (P4) and (P5) in (4), yields

$$\begin{aligned} \dot{S}(\mathbf{r}, \boldsymbol{\theta}) &= \sum_{k=1}^N \langle r_k - c_0, e^{i\theta_k} \rangle (1 - \rho_0 \omega_k + \rho_0 u_k) \\ &\quad - \sum_{k=1}^N \rho_0 \langle ie^{i\theta_k}, e^{i\theta_k} \rangle (1 - \rho_0 \omega_k + \rho_0 u_k) \end{aligned} \quad (5)$$

Since  $\langle ie^{i\theta_k}, e^{i\theta_k} \rangle = 0$ , therefore (5) is simplified to

$$\dot{S}(\mathbf{r}, \boldsymbol{\theta}) = \sum_{k=1}^N \langle r_k - c_0, e^{i\theta_k} \rangle (1 - \rho_0 \omega_k + \rho_0 u_k) \quad (6)$$

#### B. Achieving Synchronized and Balanced Collective Motion

The average linear momentum of a group of agents is a key control parameter in stabilizing their synchronized and balanced collective motion. It is maximized in synchronized collective motion and minimized in balanced collective motion. From (1a), the average linear momentum,  $p_\theta$ , of a group of  $N$ -agents, is

$$p_\theta = \frac{1}{N} \sum_{k=1}^N e^{i\theta_k} \quad (7)$$

which is also referred to as the phase order parameter [20]. The phase arrangement  $\boldsymbol{\theta}$  is synchronized if the modulus of the phase order parameter (7) equals one, that is,  $|p_\theta| = 1$ . The phase arrangement  $\boldsymbol{\theta}$  is balanced if the phase order parameter (7) equals zero, that is,  $p_\theta = 0$  [21].

The stabilization of synchronized and balanced collective motions is accomplished by considering the potential [2]

$$U(\boldsymbol{\theta}) = (N/2) |p_\theta|^2 \quad (8)$$

which reaches its unique minimum when  $p_\theta = 0$  (balanced) and its unique maximum when all phases are identical (synchronized). All other critical points of  $U$  are isolated in the shape manifold  $\mathbb{T}^N / S^1$  and are saddle points of  $U$  [2].

The time derivative of  $U$ , along the dynamics (1a), (1b) and (1c), is

$$\dot{U}(\boldsymbol{\theta}) = N \sum_{k=1}^N \left\langle p_\theta, \frac{\partial p_\theta}{\partial \theta_k} \dot{\theta}_k \right\rangle = N \sum_{k=1}^N \left\langle p_\theta, \frac{\partial p_\theta}{\partial \theta_k} (-\omega_k + u_k) \right\rangle \quad (9)$$

Using property (P4), we get

$$\dot{U}(\boldsymbol{\theta}) = \sum_{k=1}^N \left\langle p_\theta, ie^{i\theta_k} \right\rangle (-\omega_k + u_k) \quad (10)$$

#### C. Achieving Desired Angular Frequency

The agents, initially rotating around individual circles with different angular frequencies, are supposed to stabilize their collective motion on a common circle with desired angular frequency  $\Omega_0$ . To achieve the desired angular frequency  $\Omega_0$  of the common circular orbit traversed by each agent, we choose a candidate Lyapunov function

$$G(\boldsymbol{\omega}) = \frac{1}{2} \sum_{k=1}^N (\omega_k - \Omega_0)^2 \quad (11)$$

which is minimized when  $\omega_k = \Omega_0$ , for all  $k = 1, \dots, N$ .

Taking the time derivative of  $G$  along the dynamics (1a), (1b) and (1c), yields

$$\dot{G}(\boldsymbol{\omega}) = \sum_{k=1}^N (\omega_k - \Omega_0) \dot{\omega}_k = \sum_{k=1}^N (\omega_k - \Omega_0) \mu_k \quad (12)$$

#### D. Composite Lyapunov Function and Control Laws

In this subsection, by constructing the composite Lyapunov functions, the control laws  $u_k$  and  $\mu_k$  are proposed to prove the stabilization of both, balanced and synchronized collective motions (of the agents) around a common circle of desired radius  $\rho_0 = \Omega_0^{-1}$  and desired center  $c_0$ , which is fixed.

**Theorem 1:** Consider the system dynamics (1a), (1b) and (1c) with control laws

$$u_k = -\Omega_0 \left( \kappa \langle r_k - c_0, e^{i\theta_k} \rangle + K \langle p_\theta, ie^{i\theta_k} \rangle \right) \quad (13)$$

$$\mu_k = -\kappa \rho_0 (\omega_k - \Omega_0) - \Omega_0 u_k. \quad (14)$$

For  $K > 0$  and  $\kappa > 0$ , all the agents asymptotically stabilize to a clockwise circular motion of radius  $\rho_0 = \Omega_0^{-1} > 0$  about a fixed center  $c_0$  with their relative phases in balanced state.

*Proof:* Consider a composite Lyapunov function

$$V_1(\mathbf{r}, \boldsymbol{\theta}, \boldsymbol{\omega}) = \kappa S(\mathbf{r}, \boldsymbol{\theta}) + \rho_0 K U(\boldsymbol{\theta}) + \rho_0^3 G(\boldsymbol{\omega}); \quad \kappa, K > 0 \quad (15)$$

Using (6), (10) and (12), the time derivative of the Lyapunov function  $V_1$  along the dynamics (1a), (1b) and (1c), is

$$\begin{aligned} \dot{V}_1(\mathbf{r}, \boldsymbol{\theta}, \boldsymbol{\omega}) = & \kappa \sum_{k=1}^N \langle r_k - c_0, e^{i\theta_k} \rangle (1 - \rho_0 \omega_k + \rho_0 u_k) \\ & + K \sum_{k=1}^N \langle p_\theta, ie^{i\theta_k} \rangle (-\rho_0 \omega_k + \rho_0 u_k) \\ & + \rho_0^3 \sum_{k=1}^N (\omega_k - \Omega_0) \mu_k \end{aligned} \quad (16)$$

Note that

$$\sum_{k=1}^N \langle p_\theta, ie^{i\theta_k} \rangle = -\frac{1}{N} \sum_{k=1}^N \sum_{j=1}^N \sin(\theta_j - \theta_k) = 0 \quad (17)$$

Using (17), (16) can be rewritten as

$$\begin{aligned} \dot{V}_1(\mathbf{r}, \boldsymbol{\theta}, \boldsymbol{\omega}) = & \sum_{k=1}^N \left( \kappa \langle r_k - c_0, e^{i\theta_k} \rangle + K \langle p_\theta, ie^{i\theta_k} \rangle \right) \\ & \times (1 - \rho_0 \omega_k + \rho_0 u_k) + \rho_0^3 \sum_{k=1}^N (\omega_k - \Omega_0) \mu_k \end{aligned} \quad (18)$$

Under the controls in (13) and (14), the time derivative of  $V_1$  results in

$$\dot{V}_1(\mathbf{r}, \boldsymbol{\theta}, \boldsymbol{\omega}) = -\rho_0^2 \sum_{k=1}^N u_k^2 - \kappa \rho_0^4 \sum_{k=1}^N (\omega_k - \Omega_0)^2 \leq 0 \quad (19)$$

According to LaSalle's Invariance theorem [22], all the solutions of (1a), (1b) and (1c) with the controls in (13) and (14) converges to the largest invariant set contained in

$$M = \left\{ \mathbf{r}, \boldsymbol{\theta}, \boldsymbol{\omega} \mid \kappa \langle r_k - c_0, e^{i\theta_k} \rangle = -K \langle p_\theta, ie^{i\theta_k} \rangle; \omega_k = \Omega_0 \right\}.$$

for all  $k = 1, \dots, N$ . In this set, the controls in (13) and (14) evaluates to zero, that is,  $u_k = \mu_k = 0$  for all  $k = 1, \dots, N$ . It implies that  $M$  is the largest invariant set. Also, in the set  $M$ , dynamics (1b) reduce to

$$\dot{\theta}_k = -\Omega_0 \quad (20)$$

which, upon integration, yields

$$\theta_k(t) = -\Omega_0 t + \theta_k(0) = \phi_k(t) \quad (\text{say}) \quad (21)$$

Using (21), dynamics (1a) results in

$$\dot{r}_k = e^{i\phi_k(t)} \quad (22)$$

which, upon integration, yields

$$r_k(t) = r_k(0) + i\rho_0 e^{i\phi_k(t)} \quad (23)$$

Here,  $r_k(0)$  and  $\theta_k(0)$ , are constants and denote the position and orientation of the  $k$ -th agent just after stabilization, respectively. Now, we differentiate (13) with respect to time and obtain

$$\kappa \frac{d}{dt} \langle r_k - c_0, e^{i\theta_k} \rangle + K \frac{d}{dt} \langle p_\theta, ie^{i\theta_k} \rangle = 0 \quad (24)$$

Note that

$$\frac{d}{dt} \langle p_\theta, ie^{i\theta_k} \rangle = \langle -i\Omega_0 p_\theta, ie^{i\theta_k} \rangle + \langle p_\theta, -\Omega_0 e^{i\theta_k} \rangle = 0 \quad (25)$$

Substituting (21), (23) and (25) in (24), we get

$$\frac{d}{dt} \langle r_k(0) + i\rho_0 e^{i\phi_k(t)} - c_0, e^{i\phi_k(t)} \rangle = 0 \quad (26)$$

Using property (P5) along with the fact that  $\langle i\rho_0 e^{i\phi_k(t)}, e^{i\phi_k(t)} \rangle = 0$ , (26) simplifies to

$$\frac{d}{dt} \langle r_k(0) - c_0, e^{i\phi_k(t)} \rangle = 0 \quad (27)$$

Since,  $r_k(0)$  and  $c_0$  both are constants. Therefore, (27) is satisfied only if

$$r_k(0) = c_0 \quad (28)$$

for all  $k = 1, \dots, N$ . Substituting (28) in (23), yields

$$r_k(t) = c_0 + i\rho_0 e^{i\phi_k(t)} \quad (29)$$

for all  $k = 1, \dots, N$ , which is the position of the  $k$ -th agent rotating around a circle of radius  $\rho_0$  and centered at  $c_0$  (see Eq. (2)). This implies that the set of circular formations with radius  $\rho_0 = \Omega_0^{-1}$  and center  $c_0$  is asymptotically stable for  $\kappa, K > 0$ . Moreover, the potential  $U$  also approaches zero due to the non-increasing nature of  $V_1$  (since  $\dot{V}_1 \leq 0$ ). As a result, the phase arrangement of the agents, in the set  $M$ , lies in the critical set of  $U$  where  $p_\theta = 0$  (Phase balancing). It completes the proof. ■

**Theorem 2:** Consider the system dynamics (1a), (1b), and (1c) with control laws (13) and (14). For  $K < 0$  and  $\kappa > 0$ , all the agents asymptotically stabilize to a clockwise circular motion of radius  $\rho_0 = \Omega_0^{-1} > 0$  about a fixed center  $c_0$  with their relative phases in synchronized state.

*Proof:* Consider a composite Lyapunov function

$$\begin{aligned} V_2(\mathbf{r}, \boldsymbol{\theta}, \boldsymbol{\omega}) = & \kappa S(\mathbf{r}, \boldsymbol{\theta}) - \rho_0 K (N/2 - U(\boldsymbol{\theta})) \\ & + \rho_0^3 G(\boldsymbol{\omega}); \quad \kappa > 0, K < 0 \end{aligned} \quad (30)$$

Note that the magnitude of the average linear momentum  $p_\theta$  in (7) satisfies  $0 \leq |p_\theta| \leq 1$ , which ensures that  $V_2 \geq 0$ . Taking the time derivative of  $V_2$  along the dynamics (1a), (1b) and (1c), yields

$$\dot{V}_2(\mathbf{r}, \boldsymbol{\theta}, \boldsymbol{\omega}) = \dot{V}_1 = -\rho_0^2 \sum_{k=1}^N u_k^2 - \kappa \rho_0^4 \sum_{k=1}^N (\omega_k - \Omega_0)^2 \leq 0$$

Since  $\dot{V}_2 = \dot{V}_1$ , the proof follows the same steps as used to prove Theorem 1. This concludes that, for all  $k = 1, \dots, N$ , the set of circular formations with radius  $\rho_0 = \Omega_0^{-1}$  and center  $c_0$  is asymptotically stable for  $\kappa > 0, K < 0$ . Moreover, the potential  $N/2 - U$  also approaches zero due to the non-increasing nature of  $V_2$  (since  $\dot{V}_2 \leq 0$ ). As a result, the phase

arrangement of the agents, in the set  $M$ , lies in the critical set of  $N/2 - U$  where  $|p_\theta| = 1$  (Phase synchronization). It completes the proof.  $\blacksquare$

#### IV. SPLAY PHASE STABILIZATION

The splay phase is an arrangement in which the agents are uniformly distributed around the desired common circle in a way that their phases are separated by multiples of  $2\pi/N$ . The  $m$ -th harmonic of the phase order parameter  $p_\theta$ , which plays an important role in stabilizing the splay phase arrangement, is given by [1], [2]

$$p_{m\theta} = \frac{1}{mN} \sum_{k=1}^N e^{im\theta_k} \quad (31)$$

where  $m \in \mathbb{N} \triangleq \{1, 2, 3, \dots\}$ . The splay phase arrangement occurs when the condition

$$p_\theta = p_{2\theta} = \dots = p_{\lfloor N/2 \rfloor \theta} = 0 \quad (32)$$

holds [1], [2], where  $\lfloor N/2 \rfloor$  is the largest integer less than or equal to  $N/2$ . Condition (32) indicates that the splay phase arrangement corresponds to the phase balancing of the first  $\lfloor N/2 \rfloor$  harmonics of  $p_\theta$ . Therefore, in order to stabilize the splay phase arrangements, we use the potential function given as,

$$W(\boldsymbol{\theta}) = \frac{N}{2} \sum_{m=1}^{\lfloor N/2 \rfloor} |p_{m\theta}|^2 \quad (33)$$

which attains its minimum in the splay state. The time derivative of  $W$  along the dynamics (1a), (1b) and (1c), is

$$\dot{W}(\boldsymbol{\theta}) = N \sum_{k=1}^N \sum_{m=1}^{\lfloor N/2 \rfloor} \left\langle p_{m\theta}, \frac{\partial p_{m\theta}}{\partial \theta_k} (-\omega_k + u_k) \right\rangle \quad (34)$$

Using property (P4), we get

$$\dot{W}(\boldsymbol{\theta}) = \sum_{k=1}^N \sum_{m=1}^{\lfloor N/2 \rfloor} \left\langle p_{m\theta}, ie^{im\theta_k} \right\rangle (-\omega_k + u_k) \quad (35)$$

*Theorem 3:* Consider the system dynamics (1a), (1b) and (1c) with control laws

$$u_k = -\Omega_0 \left( \kappa \left\langle r_k - c_0, e^{i\theta_k} \right\rangle + K \sum_{m=1}^{\lfloor N/2 \rfloor} \left\langle p_{m\theta}, ie^{im\theta_k} \right\rangle \right) \quad (36)$$

$$\mu_k = -\kappa \rho_0 (\omega_k - \Omega_0) - \Omega_0 u_k. \quad (37)$$

For  $K > 0$  and  $\kappa > 0$ , all the agents asymptotically stabilize to a clockwise circular motion of radius  $\rho_0 = \Omega_0^{-1} > 0$  about a fixed center  $c_0$  with their relative phases in splay state.

*Proof:* Consider a composite Lyapunov function

$$V(\mathbf{r}, \boldsymbol{\theta}, \boldsymbol{\omega}) = \kappa S(\mathbf{r}, \boldsymbol{\theta}) + \rho_0 K W(\boldsymbol{\theta}) + \rho_0^3 G(\boldsymbol{\omega}); \quad \kappa, K > 0 \quad (38)$$

Using (6), (12) and (35), the time derivative of  $V$  along the dynamics (1a), (1b) and (1c), is

$$\begin{aligned} \dot{V}(\mathbf{r}, \boldsymbol{\theta}, \boldsymbol{\omega}) &= \kappa \sum_{k=1}^N \left\langle r_k - c_0, e^{i\theta_k} \right\rangle (1 - \rho_0 \omega_k + \rho_0 u_k) \\ &\quad + K \sum_{k=1}^N \left\langle p_{m\theta}, ie^{im\theta_k} \right\rangle (-\rho_0 \omega_k + \rho_0 u_k) \\ &\quad + \rho_0^3 \sum_{k=1}^N (\omega_k - \Omega_0) \mu_k \end{aligned} \quad (39)$$

Note that

$$\sum_{k=1}^N \left\langle p_{m\theta}, ie^{im\theta_k} \right\rangle = -\frac{1}{mN} \sum_{k=1}^N \sum_{j=1}^N \sin m(\theta_j - \theta_k) = 0 \quad (40)$$

Using (40), (39) can be rewritten as

$$\begin{aligned} \dot{V}(\mathbf{r}, \boldsymbol{\theta}, \boldsymbol{\omega}) &= \sum_{k=1}^N \left( \kappa \left\langle r_k - c_0, e^{i\theta_k} \right\rangle + K \left\langle p_{m\theta}, ie^{im\theta_k} \right\rangle \right) \\ &\quad \times (1 - \rho_0 \omega_k + \rho_0 u_k) + \rho_0^3 \sum_{k=1}^N (\omega_k - \Omega_0) \mu_k \end{aligned}$$

Under the controls in (36) and (37), the time derivative of  $V$  results in

$$\dot{V}(\mathbf{r}, \boldsymbol{\theta}, \boldsymbol{\omega}) = -\rho_0^2 \sum_{k=1}^N u_k^2 - \kappa \rho_0^4 \sum_{k=1}^N (\omega_k - \Omega_0)^2 \leq 0 \quad (41)$$

According to LaSalle's Invariance theorem [22], all the solutions of (1a), (1b) and (1c) with the controls in (36) and (37) converge to the largest invariant set contained in

$$M' = \left\{ \mathbf{r}, \boldsymbol{\theta}, \boldsymbol{\omega} \mid \kappa \left\langle r_k - c_0, e^{i\theta_k} \right\rangle = -K \left\langle p_{m\theta}, ie^{im\theta_k} \right\rangle; \omega_k = \Omega_0 \right\}$$

for all  $k = 1, \dots, N$ . In this set, the controls in (36) and (37) evaluates to zero, that is,  $u_k = \mu_k = 0$  for all  $k = 1, \dots, N$ . It implies that  $M'$  is the largest invariant set. Following similar analysis as used to prove Theorem 1, it is evident that, in the set  $M'$ , dynamics (1a) and (1b) give rise to (29). Therefore, it can be concluded that, for all  $k = 1, \dots, N$ , the set of circular formations with radius  $\rho_0 = \Omega_0^{-1}$  and center  $c_0$  is asymptotically stable for  $\kappa, K > 0$ . Moreover, the potential  $W$  also approaches zero due to the non-increasing nature of  $V$  (since  $\dot{V} \leq 0$ ). As a result, the phase arrangement of the agents, in the set  $M'$ , lies in the critical set of  $W$  where  $p_{m\theta} = 0$  for  $m = 1, \dots, \lfloor N/2 \rfloor$  (splay phase arrangement). This completes the proof.  $\blacksquare$

#### V. SIMULATION RESULTS

In this section, simulations are provided to validate the theoretical results obtained in the previous sections. We render the simulation results for  $N = 6$  agents where their initial positions, initial headings and initial angular frequencies are randomly generated. Figure 2 shows the stabilization of the collective motion of the agents in the clockwise direction around a common circle of desired radius  $\rho_0 = \Omega_0^{-1} = 10$  and desired center  $c_0 = (5, 5)$ . In all figures, the trajectory of centroid is shown by a dashed line.

Figures 2(a) and 2(b) depict the collective circular motions in the balanced and synchronized phase arrangements, respectively, which are obtained under the controls (13) and (14). For phase balancing, we set the controller gains as  $K = 0.5$  and  $\kappa = 0.1$ , whereas, for phase synchronization, we set  $K = -0.5$  and  $\kappa = 0.1$ . In figure 2(c), the collective motion in the splay state is shown, where the agents are spread out equidistantly on the desired common circle and have an angular separation of  $60^\circ$ . The splay formation is obtained under the controls (36) and (37) for controller gains  $K = 0.5$  and  $\kappa = 0.1$ . Note that, in the figures 2(a) and 2(c), the final position of the centroid of the group coincides with the center  $c_0 = (5, 5)$  of the common circle. It happens because of the fact that, in balanced and splay formations, the linear momentum  $p_\theta = 0$ , which causes (2) to reduce to  $c_0 = (1/N) \sum_{k=1}^N r_k$ , which is the average position of all the agents, i.e. the position of their centroid.

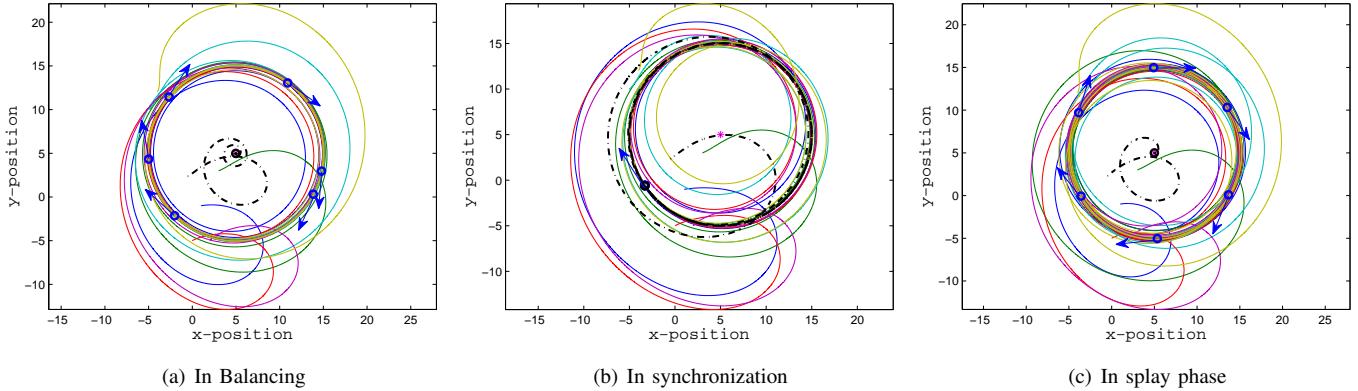


Fig. 2. Stabilization of  $N = 6$  agents on the common circle of desired radius  $\rho_0 = \Omega_0^{-1} = 10$  and desired center  $c_0 = (5, 5)$  in the symmetric patterns of their phases. (a) Under the control laws (13) and (14) with  $K = 0.5$  and  $\kappa = 0.1$ . (b) Under control laws (13) and (14) with  $K = -0.5$  and  $\kappa = 0.1$ . (c) Under control laws (36) and (37) with  $K = 0.5$  and  $\kappa = 0.1$ .

## VI. CONCLUSIONS

In this paper, a control scheme, which consists of two feedback controllers, has been proposed to stabilize the collective motion of a group of  $N$ -identical agents around a desired common circle with their phases either in synchronization, in balanced, or in splay formations. One of the controllers controls the heading and the other controller controls the angular frequency of each agent. These feedback controls have been derived from composite Lyapunov functions, which reach their minimum in the desired configuration of the collective motion. It has been shown that the closed-loop system is asymptotically stable under the action of these feedback control laws.

## VII. ACKNOWLEDGEMENTS

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# Reachability of Agents with Double Integrator Dynamics in Cyclic Pursuit

Dwaipayan Mukherjee and Debasish Ghose

**Abstract**—Some recent work on cyclic pursuit systems with double integrator dynamics has probed the stability of certain proposed laws and investigated the stability of several formations for such a system of agents. Some of these laws use the relative position information of two leading neighbors, instead of one as in case of single integrator dynamics. In some others the relative position of only one leader is used along with its relative velocity and a damping term. In this paper, a new law is proposed which guarantees stability. An algorithm is proposed which enables rendezvous of the agents at any desired point in the two-dimensional space. The gains corresponding to each agent are different and, along with their initial velocities, are considered to be the decision variables. The theoretical results are backed by simulation studies.

## I. INTRODUCTION

In conventional cyclic pursuit with single integrator dynamics, each agent  $i$  pursues its leader  $i + 1$  modulo  $n$ , where  $n$  is the total number of agents. Some results on the set of reachable points for such agents, where they may rendezvous, are available in the literature [1]. However, they consider single integrator dynamics, wherein the velocity of agent  $i$  is proportional to the distance separating the agent from its leader  $i + 1$  and is in a direction along the vector joining  $i$  and  $i + 1$ . Formation stability for such agents in cyclic pursuit have also been analyzed [2]. In [1] it has been shown that the gains can be heterogeneous and at most one of the gains can be negative (with a lower bound), thereby enabling rendezvous at almost any point in the state space, except in some diagnostic cases. Some other aspects of the consensus problem have also received attention in the literature [3]. In general, the consensus problem in multi-agent systems is important as the agents may be required to achieve identical values in the absence of a central controller to instruct them.

In [4], it has been stated that double integrator dynamics are more suited to capturing the model of Unmanned Air Vehicles (UAVs), while single integrator dynamics suffice to capture the model of a Unmanned Ground Vehicle (UGV). In [5], the existing cyclic pursuit laws for single integrator

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dynamics have been extended to consider double integrator dynamics, which necessitate the relative position and velocity information of two neighboring agents instead of one. However, the objective of [5] was to ensure stable evenly spaced circular formation or evenly spaced Archimedean spiral formations. Collective rotating motion of second order multi-agent systems are also considered in [6]. But, here also the primary interest was to investigate the stability of formations rather than achieving positional consensus. In [7]-[8], conditions on Euler angle and damping gain are investigated for an existing algorithm with double integrator dynamics, that result in rendezvous, circular motion or movement along a logarithmic spiral. Co-operative target tracking through desired formations, using a double integrator robot model, is studied in [9]. Stable vehicular formations are also studied in [10] for both single and double integrator dynamics. In particular, formation patterns for agents in cyclic pursuit (with double integrator dynamics) are probed in [11]. For agents with directed acyclic graphs, collective motion and formation patterns are investigated in [12]. In [13] deviated linear cyclic pursuit has been studied and the conditions for stability have been derived assuming both heterogeneous gains and deviations of the agents. It has been shown that using heterogeneous gains and deviations, rendezvous is possible at certain points outside the convex hull of the initial positions of the agents, which are not reachable by any other known cyclic pursuit strategy. However, not all points are reachable and the dynamics of the agents are considered to be of single integrator type.

In the present work, agents with double integrator dynamics have been considered. These agents may have different gains and initial velocities. Depending on the choice of gains and initial velocities (which are the decision variables for the proposed algorithm), it is proved that positional consensus can be achieved at any desired point in the two-dimensional space. This same goal might have been achieved by incorporating the information about the desired goal point into the control signal. But this would have implied additional information which is extrinsic to the agents, that is not directly related to the agents position or velocity. The results are proved using techniques of matrix theory, [14] that involve estimating the regions within which the eigen-values of a system may lie. The case of level flight is considered with no variation in altitude. Hence, rendezvous in two dimensions is sufficient for positional consensus.

The paper is organized in the following manner. In Section II, a control law is proposed for agents with double integrator dynamics in cyclic pursuit and the stability of the same is

investigated. Section III shows that the reachability set using this control law includes any point in the two dimensional space. A motivating example has been presented to illustrate the limitation of existing cyclic pursuit schemes (with single integrator dynamics) with respect to reachability. In Section IV, an algorithm is proposed to choose the decision variables depending on the desired point of convergence (rendezvous). In Section V, the theoretical results in the earlier sections are backed by numerical examples and corresponding simulation results. Section VI concludes the paper by suggesting some directions along which future studies may be focussed.

## II. CONTROL LAW FOR DOUBLE INTEGRATOR DYNAMICS

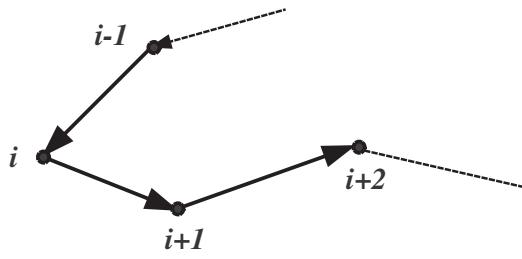


Fig. 1. The Cyclic Pursuit Scheme.

Consider the dynamics of agent  $i$  (modulo  $n$ ) given by:

$$\dot{z}_i = v_i \quad (1)$$

$$\dot{v}_i = u_i \quad (2)$$

$$u_i = k_i(z_{i+1} - z_i) - 2\sqrt{k_i}v_i; \quad k_i > 0 \quad \forall i \quad (3)$$

where,  $z_i \in \mathbb{C}$  denotes the position of agent  $i$  and  $v_i \in \mathbb{C}$  its velocity. As with conventional cyclic pursuit, where each agent has information about only one of its neighbors, (depicted in Fig. 1) here also the same scheme of information exchange holds. Since the motion along the  $x$  and  $y$  directions are decoupled and similar, it suffices to replace  $z_i$  by  $x_i \in \mathbb{R}$  and  $v_i$  by  $v_{ix} \in \mathbb{R}$  in (1)-(3), for the purpose of investigation. The same dynamics is valid along the  $y$ -direction. The system dynamics can thus be written in compact form as:

$$\dot{w} = Aw, \quad w \in \mathbb{R}^{2n}, \quad A \in \mathbb{R}^{2n \times 2n} \quad (4)$$

$$w = [x_1 \ x_2 \ \dots \ x_n \ v_{1x} \ v_{2x} \ \dots \ v_{nx}]^T \quad (5)$$

$$A = \begin{pmatrix} 0 & I_n \\ P & Q \end{pmatrix} \quad (6)$$

$$P = \begin{pmatrix} -k_1 & k_1 & \dots & 0 \\ 0 & -k_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ k_n & 0 & \dots & -k_n \end{pmatrix} \quad (7)$$

$$Q = \text{diag}(-2\sqrt{k_1} \ -2\sqrt{k_2} \ \dots \ -2\sqrt{k_n}) \quad (8)$$

where,  $\text{diag}()$  represents the elements along the diagonals of a diagonal matrix. The choice of  $-2\sqrt{k_i}$  as a damping term ensures that a system of order  $2n$  can be characterized by  $n$  parameters only and this makes the analysis

tractable. Furthermore, these  $n$  parameters (that is,  $k_i \ \forall i$ ) are sufficient to ensure rendezvous at any desired point in the two-dimensional space, as will be illustrated later. In order to investigate the stability of the control law proposed above, the characteristic polynomial corresponding to the system given by (4)-(8) needs to be studied. Let  $r(s)$  be the characteristic polynomial given by:

$$r(s) = \det(sI_{2n} - A) = \det \begin{pmatrix} sI_n & -I_n \\ -P & sI_n - Q \end{pmatrix} \quad (9)$$

Since the blocks in the first row block of  $(sI_{2n} - A)$  commute, in accordance with [15], the expression for  $r(s)$  can be written as:

$$r(s) = \det[(sI_n - Q)sI_n - PI_n] = \prod_i (s + \sqrt{k_i})^2 - \prod_i k_i \quad (10)$$

From (10), it is obvious that the characteristic equation  $r(s) = 0$  has a root at the origin. If the nullity of  $A$  is unity, it implies that the non-trivial null space of  $A$  is spanned by the vector  $[\underbrace{1 \ 1 \ \dots \ 1}_n \ \underbrace{0 \ 0 \ \dots \ 0}_n]^T$  as  $A$  has only one eigen-value at the origin. Furthermore, if all the other eigen-values of  $A$  are in the open left half plane, then, by the same arguments as presented in [1]-[2], the system is stable and positional consensus will be reached asymptotically. From the null vector, it can also be deduced that the velocities of the agents will be zero at steady state. In order to investigate the stability of  $A$ , another matrix having the same characteristic equation,  $r(s)$ , given by  $H \in \mathbb{R}^{2n \times 2n}$  is considered:

$$H = \begin{pmatrix} -\sqrt{k_1} & \sqrt{k_1} & 0 & \dots & 0 \\ 0 & -\sqrt{k_1} & \sqrt{k_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -\sqrt{k_n} & \sqrt{k_n} \\ \sqrt{k_n} & 0 & \dots & \dots & -\sqrt{k_n} \end{pmatrix} \quad (11)$$

The fact that  $H$  has the same characteristic equation,  $r(s)$ , can be verified by the principle of mathematical induction by checking for  $n = 1, 2$  and then assuming the pattern to hold for  $n = m$ , it can be checked that it holds for  $n = m + 1$ . Upon a direct application of Gershgorin's theorem, [6], as in [1], it is apparent that except for one eigen-value at the origin, all the other eigen-values of  $H$  must lie within the open left half plane. Since  $H$  and  $A$  have the same characteristic equations, their eigen-values must be the same. Thus, the matrix  $A$  must also have exactly one eigen-value at the origin and all other eigen-values in the open left half plane. Therefore, stability of the system is guaranteed with the control law proposed in (3). This ensures rendezvous. In view of the above discussion, the following theorem may now be stated.

*Theorem 1:* Consider the system given by (4)-(8), with the control law as in (3). The system has exactly one eigen-value at the origin and all other eigen-values in the open left half plane, thereby ensuring stability and positional consensus of the multi-agent system.

*Proof:* Consider the matrix  $H$  in (11). Corresponding to each row of  $H$ , a Gershgorin disc may be constructed with

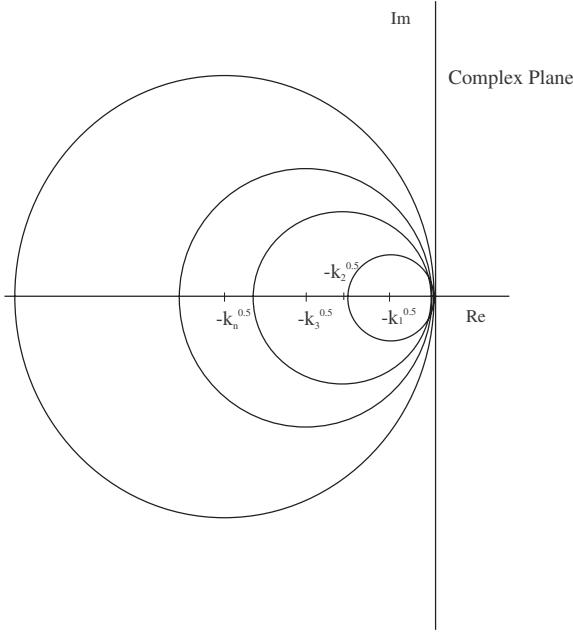


Fig. 2. Gershgorin discs corresponding to each row of  $H$ .

its center at  $(-\sqrt{k_i}, 0)$  and radius equal to  $\sqrt{k_i}$  as shown in Fig. 2. Since each gain  $k_i$  appears in two rows, there are effectively  $n$  distinct discs at most (assuming all the gains are different). Now, by Gershgorin's Theorem [14], all the eigen-values of the matrix  $H$  must lie within the union of these discs. Since all these discs lie within the left half plane (except for the origin), the eigen-values of  $H$  must be in the left half plane too and thus the characteristic equation of  $H$  must have all its roots in the same region. This same characteristic equation  $r(s)$  is shared by the matrix  $A$  and so the same conclusions can be drawn about the eigen-values of  $A$ . An inspection of  $r'(s)$ , (the derivative of  $r(s)$  with respect to  $s$ ) reveals that it does not have a root at zero. Thus  $r(s)$  has only one root at the origin and no repeated roots there. According to [16], each co-ordinate of the solution to the initial value problem (4) is a linear combination of functions of the form  $t^k e^{a_i t} \cos b_i t$  or  $t^k e^{a_i t} \sin b_i t$ , where  $\lambda_i = a_i + jb_i$  is an eigen-value of the matrix  $A$ , and  $k$  ( $0 \leq k \leq n-1$ ) is the algebraic multiplicity of the eigen-value  $\lambda_i$ . Thus, all these components will decay at steady state because  $a_i < 0 \ \forall i$ , except for the component corresponding to the zero eigen-value. Hence, the null vector (eigen-vector corresponding to the zero eigen-value) determines the steady state behaviour. From the null vector  $[\underbrace{1 \ 1 \ \dots \ 1}_n \ \underbrace{0 \ 0 \ \dots \ 0}_n]^T$ , it is clear that

at steady state positional consensus is achieved with zero velocity. This is also in accordance with the results in [1], [2].  $\blacksquare$

### III. ANALYSIS OF REACHABLE SET

The following example illustrates the limitations of heterogeneous cyclic pursuit [1], with single integrator dynamics,

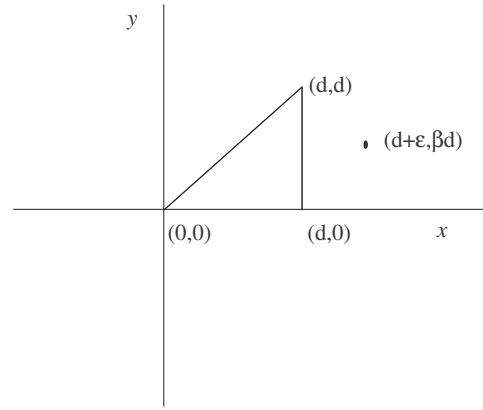


Fig. 3. Desired Rendezvous point for single integrator dynamics.

in terms of reachability.

*A Motivating Example:* Consider 3 agents starting from the vertices of an isosceles right angled triangle at  $[(0,0), (d,0), (d,d)]$  as shown in Fig. 3. Suppose the desired rendezvous point is such that its abscissa is greater than  $d$  and its ordinate is between 0 and  $d$ . Let the gains for the agents be  $k_i$ . The co-ordinates of the rendezvous point may be written as  $(d + \epsilon, \beta d)$ , where  $\epsilon > 0$  is any positive real number and  $0 < \beta < 1$ . Suppose that  $\sum_i \frac{1}{k_i} = p$ , without loss of generality. This quantity  $p$  may be suitably chosen to scale up or scale down the gains. It is known from [1] that the point of convergence of the agents in cyclic pursuit is given by

$$Z_f = \frac{\sum_{i=1}^n \frac{z_{i0}}{k_i}}{\sum_{p=1}^n \frac{1}{k_p}}.$$

where  $Z_f$  is the co-ordinate of the rendezvous point denoted by a complex number and  $z_{i0}$  is similarly the initial position of agent  $i$ . In accordance with this, the following equations may now be written for rendezvous of the agents at the desired location:

$$\begin{aligned} \frac{d}{k_2} + \frac{d}{k_3} &= (d + \epsilon)p \\ \frac{d}{k_3} &= \beta dp \end{aligned}$$

It is obvious that the values of  $\frac{1}{k_i}$  are given by

$$\begin{aligned} \frac{1}{k_1} &= -\frac{p\epsilon}{d} \\ \frac{1}{k_2} &= p(1 + \frac{\epsilon}{d} - \beta) \\ \frac{1}{k_3} &= p\beta. \end{aligned}$$

Now, depending on the sign of  $p$ , two cases may arise. If  $p < 0$ , two of the gains,  $k_2$  and  $k_3$ , become negative, thereby violating the conditions for stability and ruling out rendezvous [1]. On the other hand, if  $p > 0$ , only  $k_1 < 0$ , but multiplying both sides of the inequality,  $\sum_i \frac{1}{k_i} > 0$ , by the negative quantity  $k_1 k_2 k_3$ , it is easy to see that  $k_1 k_2 + k_2 k_3 +$

$k_3 k_1 < 0$ . According to [1], in the third order characteristic equation corresponding to this system, the coefficient of  $s$  is  $k_1 k_2 + k_2 k_3 + k_3 k_1$ . Therefore, the last inequality again implies instability and consequent failure of the agents to converge to a point. Thus, it may be concluded that no known cyclic pursuit scheme with single integrator dynamics can ensure rendezvous at the desired point. The deviated cyclic pursuit scheme presented in [13] does enable rendezvous at certain points outside the convex hull of the initial co-ordinates, but the distance of the point of convergence from the convex hull is restricted by the limits on the permissible deviations discussed therein.  $\square$

From (1)-(3), the following may be written after a simple algebraic manipulation (replacing  $z_i \in \mathbb{C}$  by  $x_i \in \mathbb{R}$ ):

$$\frac{\ddot{x}_i + 2\sqrt{k_i}\dot{x}_i}{k_i} = x_{i+1} - x_i \quad (12)$$

Upon summation of both sides, with index determined as modulo  $n$ , the following expression is obtained:

$$\sum_{i=1}^n \frac{\ddot{x}_i + 2\sqrt{k_i}\dot{x}_i}{k_i} = 0 \quad (13)$$

Integrating (13) over the interval  $t_0$  (initial time) to  $t_f$  (final time), it is apparent that:

$$\sum_{i=1}^n \frac{\dot{x}_i(t_f) + 2\sqrt{k_i}x_i(t_f)}{k_i} = \sum_{i=1}^n \frac{\dot{x}_i(t_0) + 2\sqrt{k_i}x_i(t_0)}{k_i} \quad (14)$$

Now, at  $t = t_f$ ,  $\dot{x}_i(t_f) = 0$ ,  $\forall i$ , and  $x_i(t_f) = X_f$ ,  $\forall i$ , as are evident from the null vector obtained in the previous section. Thus, the point of convergence is given by:

$$X_f = \frac{\sum_{i=1}^n \frac{\dot{x}_i(t_0) + 2\sqrt{k_i}x_i(t_0)}{k_i}}{\sum_{i=1}^n \frac{2}{\sqrt{k_i}}} \quad (15)$$

It may be noted that this point of convergence is a sum of two terms, one of which is a point inside the convex hull of the initial positions of the agents ( $\sum_{i=1}^n \frac{x_i(t_0)}{\sqrt{k_i}} / \sum_{i=1}^n \frac{1}{\sqrt{k_i}}$ ) and the other term ( $\sum_{i=1}^n \frac{\dot{x}_i(t_0)}{2k_i} / \sum_{i=1}^n \frac{1}{\sqrt{k_i}}$ ) is a function of the initial velocities of the agents and the gains chosen. If the initial velocity of at least one agent may be chosen at will, it is clear that  $X_f$  may assume any value. Hence, it is this second term that enables global reachability. The following theorem may now be stated.

**Theorem 2:** Consider the system given by (4)-(8), with the control law as in (3). The system of agents can be made to converge to any point in the two-dimensional space using a suitable choice of gains,  $k_i$  and initial velocities,  $\dot{x}_i(t_0)$  and  $\dot{y}_i(t_0)$ . The point of convergence ( $X_f, Y_f$ ) is given by:

$$X_f = \frac{\sum_{i=1}^n \frac{x_i(t_0)}{\sqrt{k_i}}}{\sum_{i=1}^n \frac{1}{\sqrt{k_i}}} + \frac{\sum_{i=1}^n \frac{\dot{x}_i(t_0)}{2k_i}}{\sum_{i=1}^n \frac{1}{\sqrt{k_i}}} \quad (16)$$

$$Y_f = \frac{\sum_{i=1}^n \frac{y_i(t_0)}{\sqrt{k_i}}}{\sum_{i=1}^n \frac{1}{\sqrt{k_i}}} + \frac{\sum_{i=1}^n \frac{\dot{y}_i(t_0)}{2k_i}}{\sum_{i=1}^n \frac{1}{\sqrt{k_i}}} \quad (17)$$

*Proof:* From Theorem 1, rendezvous of the agents is guaranteed. Thus, the point of convergence derived above

is the point where the agents rendezvous. It only remains to be shown that the entire two dimensional space can be reached. Consider a desired point  $(X_d, Y_d)$  where the agents must rendezvous. Let the point of convergence with zero initial velocities of all agents be given by  $(X_1, Y_1)$ , where,  $X_1 = \sum_{i=1}^n \frac{x_i(t_0)}{\sqrt{k_i}} / \sum_{i=1}^n \frac{1}{\sqrt{k_i}}$  and  $Y_1 = \sum_{i=1}^n \frac{y_i(t_0)}{\sqrt{k_i}} / \sum_{i=1}^n \frac{1}{\sqrt{k_i}}$ . Considering only the motion along the  $x$ -direction (the same reasoning can be extended along the  $y$ -direction), let the initial velocities of all but one agent be zero. The agent  $l$  with non-zero initial velocity may have an initial velocity given by  $(X_d - X_1) \times \sum_{i=1}^n \frac{1}{\sqrt{k_i}} \times 2k_l$ . Plugging this expression for  $\dot{x}_l(t_0)$  in (16), it is easy to see that  $X_f = X_d$ . This completes the proof.  $\blacksquare$

#### IV. AN ALGORITHM TO ENSURE GLOBAL REACHABILITY

In the previous section it was shown that any point on the two dimensional space may be reached provided at least one of the agents has a suitably chosen non-zero initial velocity. However, in practice, the agents generally start from rest and if they have a non-zero velocity then their position will vary in accordance with the governing equations. Thus, the initial position may not be as desired. One way of circumventing this problem is to give one of the agents a dithering motion prior to the start of the cyclic pursuit phase. It is proposed to split the reachability problem into two phases. In the first phase, one of the agents executes small oscillations of suitably chosen frequency about its mean position. In the next phase the cyclic pursuit law is executed, so as to rendezvous at a desired location. The algorithmic steps are outlined below:

- 1) Pick up an agent whose initial position  $Z_{i0}$  (given by  $(X_{i0}, Y_{i0})$ ) is closest to the desired point  $Z_f$  (given by  $(X_f, Y_f)$ ) of convergence, that is,  $i$  corresponding to which  $\|Z_{i0} - Z_f\|_2$  is minimum. Suppose, this corresponds to agent  $l$ , without any loss of generality.
- 2) Assuming the initial velocities to be zero, the set of gains may be chosen to rendezvous at a point  $(X_1, Y_1)$ , arbitrarily close to the initial co-ordinates of agent  $l$ . Clearly this choice is non-unique, so the gains may be suitably scaled by the designer. As a good thumb rule,  $\sum_i \frac{1}{\sqrt{k_i}} = 1$  may be an imposed constraint. Of course, any other value of the sum would have served the purpose too.
- 3) Next, the initial velocity of agent  $l$  needs to be chosen. Suppose the velocity of agent  $l$  along the  $x$  direction, prior to the initiation of the cyclic pursuit scheme, is given by:

$$v_{lx} = A_{lx} \sin(\omega_{lx} t + \phi_{lx}) \quad (18)$$

This is chosen to be a sinusoid to ensure that the agent  $l$  does not drift away from its initial position. Rather, it executes a simple harmonic motion about its mean position. A similar choice is made for velocity along the  $y$  direction.

- 4) Depending on the permissible level of oscillations in position, the choice of the ratio  $A_{lx}/\omega_{lx}$  is made,

because this ratio corresponds to the maximum shift in position for the agent  $l$ , about its mean position.

5) The parameters  $A_{lx}$ ,  $\omega_{lx}$  and  $\phi_{lx}$  (correspondingly  $A_{ly}$ ,  $\omega_{ly}$  and  $\phi_{ly}$ ) need to be chosen. At  $t = t_0$ , the velocity must be given by

$$A_{lx} \sin(\omega_{lx} t_0 + \phi_{lx}) = (X_f - X_1) \times \sum_{i=1}^n \frac{1}{\sqrt{k_i}} \times 2k_l. \quad (19)$$

Thus, for a given  $t_0$  (starting instant),  $A_{lx}$ ,  $\omega_{lx}$  and  $\phi_{lx}$  may be chosen to satisfy (19) and ensure small oscillations about the mean position of agent  $l$ .

It should be noted that this algorithm works with  $n+6$  decision variables ( $n$  gains and three terms each corresponding to velocities along  $x$  and  $y$  directions). But the system order is  $2n$  with  $n$  agent positions and  $n$  agent velocities corresponding to the  $n$  agents. The terms  $\phi_{lx}$  and  $\phi_{ly}$  ensure that even if  $\sin(\omega_{lx} t_0)$  or  $\sin(\omega_{ly} t_0)$  are zeros, the initial velocity of agent  $l$  does not become zero. This could also be ensured by suitably modifying  $\omega_{lx}$  or  $\omega_{ly}$  as per requirement, but the additional terms  $\phi_{lx}$  and  $\phi_{ly}$  enable an independent choice of the ratio  $A_{lx}/\omega_{lx}$  to ensure small oscillations.

## V. SIMULATION RESULTS

*Example 1:* Consider a system of four agents starting from the vertices of a square  $(0,0), (3,0), (3,3), (0,3)$ . It is desired that the agents should rendezvous at the point  $(10,7)$ . According to the first step of the algorithm outlined in the previous section, agent 3 is closest to the desired point of convergence. Now, the gains are chosen assuming zero initial velocities. Thus, the point  $(X_1, Y_1)$  of step 2 may be chosen as  $(2.9, 2.9)$ . One choice of gains is  $[3600 \ 3600 \ 1.108 \ 3600]$ , which results in  $\sum_i \frac{1}{\sqrt{k_i}} = 1$ . As pointed out in the previous section, the choice of the gains is not unique and the particular choice in this example is not sacrosanct. Now, agent 3 must execute a sinusoidal motion about its initial position  $(3,3)$  prior to the initiation of the cyclic pursuit phase while the other agents are stationary. If the amplitude of oscillation is restricted to 0.1 for agent 3, the ratio of amplitude to frequency ( $A_{3x}/\omega_{3x}$  and  $A_{3y}/\omega_{3y}$ ) must be less than 0.1. If  $t_0 = 1s$ , it is clear that  $A_{3x} = 15.734$ ,  $A_{3y} = 9.086$ ,  $\omega_{3x} = \omega_{3y} = 200$  and  $\phi_{3x} = \phi_{3y} = \pi/2 - 200$  satisfy (19). This particular choice also ensures that at  $t = t_0$ , the position of agent 3 is  $(3,3)$ . The additional terms  $\phi_{3x}$  and  $\phi_{3y}$  have been used to this end. Fig. 4 shows the trajectories of the four agents which converge at the desired point. The square shown in bold lines is the convex hull of the initial positions of the agents. Fig. 5 and Fig. 6 show that after 1s the cyclic pursuit is initiated and prior to that only agent 3 executes small oscillations about its mean position. It may be noted that even with positive gains, it is possible to converge outside the convex hull of the initial co-ordinates of the agents.

*Example 2:* In this example all conditions and design parameters are kept the same as in example 1 except the value of  $A_{3y}$ , which is changed to  $-2.437$ . This results in a rendezvous at the point  $(10,1.8)$ , as shown in Fig. 7. It may be noted that this point will not be reachable using conventional cyclic pursuit with single integrator dynamics,

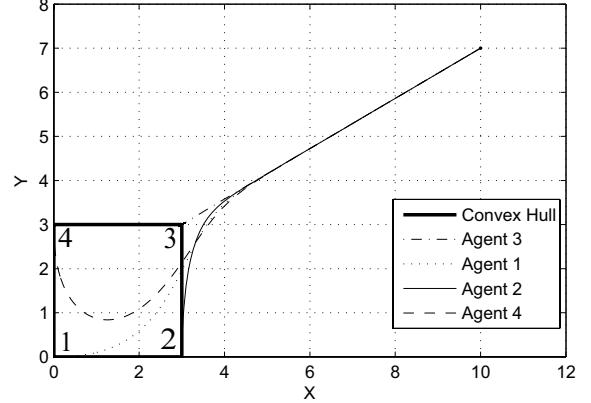


Fig. 4. Agents converging to a desired point in Example 1.

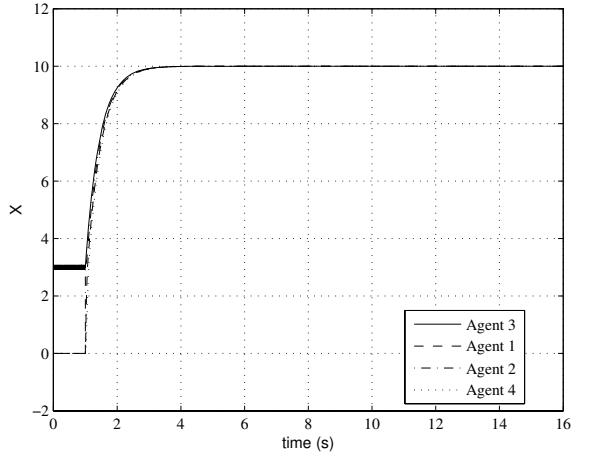


Fig. 5. Consensus in X co-ordinates for Example 1.

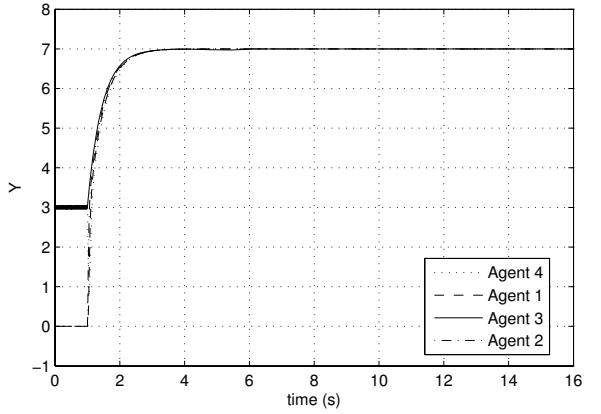


Fig. 6. Consensus in Y co-ordinates for Example 1.

even if a negative gain is used as in [1]. This is because this point does not belong to the union of the convex hull of initial co-ordinates and the cones formed by them as defined in [1]. However, the present scheme enables rendezvous at any point.

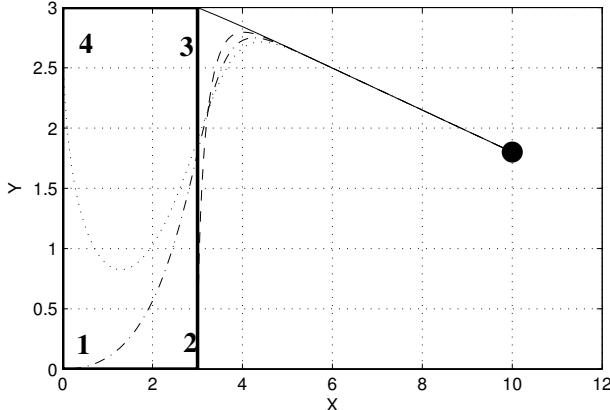


Fig. 7. Agents converging to a point outside convex hull or cones as defined in [1] in Example 2.

## VI. CONCLUSIONS

In this work, agents with double integrator dynamics in cyclic pursuit have been analyzed and a control law has been proposed. The stability of the proposed law has been proved. The expansion of the reachable set has been studied and demonstrated through simulation results. It is proved that any point in the two-dimensional space is reachable for arbitrary initial configuration of agents. A two step algorithm has been proposed to reach arbitrary locations in the two dimensional space. In future, research may be directed towards converging to a desired trajectory, where, the point of convergence may be generalized to a predetermined trajectory.

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